1. Strategy: to prove that the three lines meet at a point, we will show that \( \frac{AP}{PB} \cdot \frac{BR}{RC} \cdot \frac{CQ}{QA} = 1 \). If this is true, Ceva’s theorem says that the lines are concurrent.

Now, \( \overline{AR} \) is an angle bisector, so the Angle Bisector Theorem says \( \frac{BR}{AB} = \frac{RC}{AC} \). Then,

\[
\frac{BR}{RC} = \frac{AB}{AC} \quad (\ast)
\]

Also, \( \overline{BQ} \) is an angle bisector, so the Angle Bisector Theorem says \( \frac{QA}{AB} = \frac{CQ}{BC} \). Then,

\[
\frac{CQ}{QA} = \frac{BC}{AB} \quad (\ast)
\]

Finally, \( \overline{CP} \) is an angle bisector, so the Angle Bisector Theorem says \( \frac{AP}{CA} = \frac{PB}{BC} \). Then,

\[
\frac{AP}{BP} = \frac{AC}{BC} \quad (\ast)
\]

So, going back to what we want to show,

\[
\frac{AP}{PB} \cdot \frac{BR}{RC} \cdot \frac{CQ}{QA} = \frac{AC}{AB} \cdot \frac{AB}{BC} \cdot \frac{BC}{AB} = 1
\]

(First we substitute each fraction using (\ast) above, and then everything cancels and you get 1). So, by the second part of Ceva’s theorems, the three lines must meet at a single point.

2. Following the suggestion, let’s call \( P \) the point where \( \overline{AC} \) intersects \( \ell \). Now consider first \( \triangle ABC \) and line \( \ell \). This line intersects the three sides of \( \triangle ABC \) at \( P, K, \) and \( L \), so Menelaus says,

\[
\frac{AK}{KB} \cdot \frac{BL}{LC} \cdot \frac{CP}{PA} = 1.
\]

Now consider \( \triangle ACD \). Line \( \ell \) intersects its sides at \( P, N, \) and \( M \), so Menelaus says

\[
\frac{AP}{PC} \cdot \frac{CM}{MD} \cdot \frac{DN}{NA} = 1.
\]
Now just multiply the above equations by each other to get
\[ \frac{AK}{KB} \frac{BL}{LC} \frac{CP}{PA} \frac{AP}{PC} \frac{CM}{MD} \frac{DN}{NA} = 1, \]
and after simplifying the left-hand side we get what we wanted to prove.

3. This one is more complicated, so go slowly and make sure you have the figure next to you.

Step (a): consider \( \triangle ABC \) and the line \( QRP \) that intersects its sides at \( Q \), \( R \), and \( P \) (if you can’t see this in the crowded figure, draw a separate figure with only \( \triangle ABC \) and \( QRP \)). Then, Menelaus says
\[ \frac{AP}{PB} \frac{BR}{RC} \frac{CQ}{QA} = 1. \quad (\ast) \]

Step (b): To prove that \( \triangle ABQ \sim \triangle CMQ \) we’ll use the AA similarity principle. First, \( \angle BQA \) and \( \angle CQM \) are congruent because they’re vertical angles. Second, \( AB \parallel CD \) because they’re the sides of a square, so \( \angle BAQ \cong \angle MCQ \) because they’re alternate interior. So, the two triangles are similar, so their sides are proportional:
\[ \frac{CM}{AB} = \frac{CQ}{QA} = \frac{MQ}{BQ}. \quad (\ast\ast) \]

Step (c): We are given that \( AB \sim BP \). Then, \( AP = AB + BP = 2 \cdot BP \). From here it follows that \( \frac{AP}{PB} = 2 \) (save this for later).

Step (d): We are given that \( M \) is the midpoint of \( CD \). Now, \( CD \sim AB \) since \( ABCD \) is a square, so \( CM = \frac{1}{2} CD = \frac{1}{2} AB \). So, \( \frac{CM}{AB} = \frac{1}{2} \). Combine this with the first equality in \( (\ast\ast) \) to get \( \frac{CQ}{QA} = \frac{1}{2} \).

Finally, plug in what we found in steps (c) and (d) into \( (\ast) \). It will look like
\[ 2 \cdot \frac{BR}{RC} \cdot \frac{1}{2} = 1. \]

After canceling in the left-hand side, we get \( \frac{BR}{RC} = 1 \), so \( BR = RC \), as we wanted to prove.
4. Step (a): We are going to apply Ceva’s theorem to $\triangle PQC$ and the three segments $\overline{PB}$, $\overline{QD}$, and $\overline{CX}$ (here it may be helpful to draw the triangle and those three segments alone). The segments all meet at a point, namely $A$, so Ceva’s Theorem says

$$\frac{PX}{XQ} \cdot \frac{QB}{BC} \cdot \frac{CD}{DP} = 1.$$  

Solve this equation for $\frac{PX}{XQ}$ by multiplying both sides by the reciprocal of the other fractions to get

$$\frac{PX}{XQ} = \frac{BC}{QB} \cdot \frac{DP}{CD}.$$  

Step (b): Now we are going to apply Menelaus’ theorem to $\triangle PQC$ and line $DBY$ (again, it may be helpful to draw the triangle and that line alone). The line intersects the three sides of the triangle at $D$, $B$, and $Y$, so Menelaus’ Theorem says

$$\frac{PY}{YQ} \cdot \frac{QB}{BC} \cdot \frac{CD}{DP} = 1.$$  

Solve this equation for $\frac{PY}{YQ}$ by multiplying both sides by the reciprocal of the other fractions to get

$$\frac{PY}{YQ} = \frac{BC}{QB} \cdot \frac{DP}{CD}.$$  

Now look ($\ast$) and ($\ast\ast$) carefully until you realize that from them it follows that $\frac{PX}{XQ} = \frac{PY}{YQ}$ (because they both equal the same thing), and this is what we wanted to prove.

5. This problems requires two applications of Menelaus’ Theorem to different triangles and lines. As before, it may be helpful to draw a separate diagram for each application, consisting only the required ingredients.

Step (a) Line $\overline{AQM}$ intersects the three sides of $\triangle XBP$, so Menelaus says

$$\frac{XM}{MB} \cdot \frac{BA}{AP} \cdot \frac{PQ}{QX} = 1.$$  

Step (b) Line $\overline{AQM}$ intersects the extension of the three sides of $\triangle XCR$, so Menelaus says

$$\frac{XM}{MC} \cdot \frac{CA}{AR} \cdot \frac{RQ}{QX} = 1.$$
So, both expressions on the left-hand sides are equal to one, so they are equal to each other:

\[
\frac{XM}{MB} \cdot \frac{BA}{AP} \cdot \frac{PQ}{QX} = \frac{XM}{MC} \cdot \frac{CA}{AR} \cdot \frac{RQ}{QX}.
\]

Now, on the right-hand side, replace \(AR\) with \(AP\) (they are the same, it’s given) and also replace \(MC\) with \(BM\) (they are the same since \(M\) is the midpoint of \(AC\)). We get

\[
\frac{XM}{MB} \cdot \frac{BA}{AP} \cdot \frac{PQ}{QX} = \frac{XM}{BM} \cdot \frac{CA}{AP} \cdot \frac{RQ}{QX}.
\]

Next cancel the factors that are common to both sides, i.e. \(XM, MB, AP,\) and \(QX\), to get

\[
BA \cdot PQ = CA \cdot RQ
\]

and finally write this as a proportion to get what we wanted to prove.