Hilbert scheme of rational curves on generic quintic 3-folds

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April 8, 2022

Abstract

Let X be a generic quintic threefold in projective space $\mathbf{P}^4$ over the complex numbers. For a natural number d, let $\mathscr{M}_d(X)$ be the subscheme of $Hilb(X)$ that parametrizes irreducible rational curves of degree d on X. In this paper, we show that

(1) $\mathscr{M}_d(X)$ is smooth and of dimension 0,

(2) furthermore it consists of immersed rational curves.

(3) Parts (1) and (2) have an implication in complex geometry: if $[C] \in \mathscr{M}_d(X)$ and $c : \mathbf{P}^1 \to C$ is the normalization, the normal bundle is isomorphic to

$$\mathscr{O}_{\mathbf{P}^1}(-1) \oplus \mathscr{O}_{\mathbf{P}^1}(-1).$$

The implication is the main statement of Clemens' conjecture.

Contents

Key words: Quintic 3-folds, Rational curves, Clemens' conjecture
2020 Mathematics subject classification : 14H15, 14J30

1 Introduction

1.1 Statement

The result addresses a moduli problem in algebraic geometry over the complex numbers. The following is the statement.

Theorem 1.1.
Let X be a generic quintic threefold in $\mathbf{P}^4$ over $\mathbb{C}$. Let $\mathscr{M}_d(X)$ be the subscheme of $Hilb(X)$ parameterizing irreducible rational curves of degree d on X. Then
(1) $\mathscr{M}_d(X)$ is smooth and of dimension 0,
(2) $\mathscr{M}_d(X)$ consists of immersed rational curves. Precisely, if

$$c : \mathbf{P}^1 \to C$$

is the normalization of the image $C = c(\mathbf{P}^1)$ with the point $[C] \in \mathscr{M}_d(X)$, then c is an immersion.

Remark Various notions of the genericity are used for brevity. Precisely we'll use the expression "generic $v \in V$" and "generic with respect to w" to indicate v is a closed point in a unspecified Zariski open subset of a component of the scheme V and w is independent of the open set. Also the ambient scheme V may be omitted in the context. For instance, in the theorem 1.1, the V is the variety $\mathbf{P}H^0(\mathscr{O}_{\mathbf{P}^4}(5))$.

Part (1) relates the tangent space of the Hilbert scheme to the normal sheaf of rational curves. It leads to a group isomorphism,

$$Hom(\mathscr{I}_C/\mathscr{I}_C^2, \mathscr{O}_C) = T_{[C]}\mathscr{M}_d(X) = 0$$

where $\mathscr{I}_C$ is the ideal sheaf of the rational curve C. Now the part (2) further implies that the pullback of the normal sheaf

$$N_{c/X} := c^*\left(\mathscr{H}om(\mathscr{I}_C/\mathscr{I}_C^2, \mathscr{O}_C)\right)$$

is the vector bundle of the immersion and the bundle does not admit a non-zero section. Since this vector bundle by the adjuntion formula has degree -2, it is isomorphic to

$$\mathscr{O}_{\mathbf{P}^1}(-1) \oplus \mathscr{O}_{\mathbf{P}^1}(-1). \tag{1.1}$$

So Theorem 1.1 implies

Corollary 1.2. *The formula (1.1) is correct for all rational curves on a very general X, where the "very general X" is referred to as a quintic in the intersection of countably many determined Zariski open sets of* $\mathbf{P}H^0(\mathscr{O}_{\mathbf{P}^4}(5))$.

The corollary implies the main statement of Clemens' conjecture ([1]) which predicts that if C is smooth, there is a complex analytic formula on the normal bundle $N_{C/X} := \frac{TX|_C}{TC}$,

$$N_{C/X} \simeq \mathscr{O}_{\mathbf{P}^1}(-1) \oplus \mathscr{O}_{\mathbf{P}^1}(-1). \tag{1.2}$$

The corollary now is optimal since it is known that singular rational curves on X do exist ([5]).

1.2 Outline of the proof

1.2.1 Setting in non-moduli view

The existence of $\mathscr{M}_d(X)$ was proved in [4]. So it suffices to prove its property which is in the standard field of algebraic geometry. Our idea, however, is to go around the standard formulation to attack its concrete objects[1] represented by the moduli. This leads us to a different field. What follows is this non-moduli approach. It begins with the affine space

$$M = \bigoplus_5 H^0(\mathscr{O}_{\mathbf{P}^1}(d)),$$

the collection of 5-tuples of homogeneous polynomials in two variables of degree d. Let M_d be the open subset such that the projectivization satisfies

$$\mathbf{P}(M_d) \simeq \{c \in Hom_{bir}(\mathbf{P}^1, \mathbf{P}^4) : deg(c(\mathbf{P}^1)) = d\}. \tag{1.3}$$

To simplify the notation, we denote an element in M_d and its corresponding birational-to-image map $\mathbf{P}^1 \to \mathbf{P}^4$ by the same letter c. The upper case C denotes the image $c(\mathbf{P}^1)$, and all three are referred to as the rational curve of the element $c \in M_d$. However the difference in representation should be noticed. For instance, there is the $GL(2)$ action on M_d, induced from the automorphisms of $\mathbf{P}^1$ for each $c \in M_d$. The $\mathbf{P}(M_d)$ is obtained by modding out the group action of the 1-dimensional torus $\mathbb{G}_m$. Furthermore the Hilbert scheme $\mathscr{M}_d(\mathbf{P}^4)$

[1]The word "object" is loosely used in a general sense as a mathematical structure is a representation of various concrete objects. For instance, a geometric structure represents coordinates' charts; a moduli space represents families of objects in algebraic geometry, etc.

parametrizing irreducible rational curves $\subset \mathbf{P}^4$ of degree d can be obtained by modding out the $GL(2)$ action [2]. In the following we introduce the "objects" of the setting. Let $S = \mathbf{P}H^0(\mathscr{O}_{\mathbf{P}^4}(5))$ the projectivization of the space of quintic polynomials and S° an affine open set. Let $\mathbb{A} \subset \mathbf{P}^1$ be an affine open set of $\mathbf{P}^1$ that consists of all finite numbers in the projective plane. For a $t_0 \in \mathbb{A}$, a rational curve c gives a holomorphic map

$$\begin{array}{ccc} M_d & \rightarrow & \mathbb{A}^5 \quad (\textit{i.e. the } 5\text{-tuple } \textit{in } M_d) \\ c & \rightarrow & c(t_0), \end{array} \tag{1.4}$$

where $\mathbf{P}(\mathbb{A}^5) = \mathbf{P}^4$. The quintic $f \in S^\circ$ determines another holomorphic map

$$\begin{array}{ccc} \mathbb{A}^5 & \rightarrow & \mathbb{C} \\ z & \rightarrow & f(z). \end{array} \tag{1.5}$$

Hence the composition is a holomorphic map

$$\begin{array}{ccc} S^\circ \times M_d & \rightarrow & \mathbb{C} \\ (f, c) & \rightarrow & f(c(t_0)). \end{array} \tag{1.6}$$

Choose $5d+1$ distinct points $t_i \in \mathbb{A}$, denoted by

$$\mathbf{t} = (t_1, t_2, \cdots, t_{5d+1}) \in \prod_{5d+1} \mathbf{P}^1.$$

We obtain a holomorphic map for a fixed $f \in S^\circ$:

$$\begin{array}{cccc} \nu_0 : & M_d & \rightarrow & \mathbb{C}^{5d+1} \\ & c & \rightarrow & \Big(f(c(t_1), \cdots, f(c(t_{5d+1})) \Big). \end{array} \tag{1.7}$$

Notice the degree of the polynomial $f(c(t))$ for the variable $t \in \mathbb{A}$ is $5d$, where the polynomial $f(c(t))$ is also canonically extended to the section

$$c^*(f) \in H^0(\mathscr{O}_{\mathbf{P}^1}(5d)).$$

Hence a rational curve is represented by a point $[C] \in \mathscr{M}_d(X)$ with

$$X = div(f) \subset \mathbf{P}^4$$

if and only if C is $c(\mathbf{P}^1)$ for

$$c \in \nu_0^{-1}(\{0\}). \tag{1.8}$$

We call $\nu_0^{-1}(\{0\})$ and M_d the non-moduli objects of $\mathscr{M}_d(X)$ and $\mathscr{M}_d(\mathbf{P}^4)$ respectively, and the type of setting based on them a non-moduli view. In a non-moduli view, Theorem 1.1 claims that for a generic quintic f, $\nu_0^{-1}(\{0\})$ is smooth of dimension 4, i.e the differential map $\mathbf{d}\nu_0$ is surjective at $\nu_0^{-1}(\{0\})$.

[2] The non-moduli space M_d contains the information of $GL(2)$ action which has a profound impact on the moduli problem of Theorem 1.1.

The original problem in algebraic geometry is now converted to that in differential geometry[3]. Nonetheless the differential map has the structural obstacle – the genericity of the quintic 3-fold f. Our idea is to find a way in differentiation to evade the 3-fold f.

Such an evasion comes from a two-step-reduction: 1) variation of f in a 2-dimensional plane, 2) followed by a projection. First, let $\mathbb{P} \subset S$ be a 2-dimensional plane generic in the Grassmannian of S. In the second step, we observe the diagram of projections P_l and P_r,

$$\begin{array}{ccc} & \Gamma_{\mathbb{P}} & \\ {}^{P_l}\swarrow & & \searrow^{P_r} \\ \mathbb{P} & & M_d \end{array} \tag{1.9}$$

where $\Gamma_{\mathbb{P}}$ is the union of the irreducible components of the incidence scheme

$$\{(f, c) \in \mathbb{P} \times M_d : C \subset div(f)\}$$

such that each component dominates $\mathbb{P}$. For a subvariety $W \subset \mathbb{P}$, let $\Gamma_{\mathbb{P}} \cap (W \times M_d)$ be the intersection scheme, and J_W an irreducible component dominating W. We'll use I_W to denote the scheme-theoretical image $P_r(J_W)$. In particular $I_{\{f\}}$ for a point $f \in \mathbb{P}$ is abbreviated as I_f, and I_f is reduced to a component of $\mathscr{M}_d(X)$ for $X = div(f)$. It will be proved in Proposition 2.7 that the projection P_r is a local isomorphism to its image around a generic point. In particular, there is an isomorphism

$$T_{(f_g, c_g)}\Gamma_{\mathbb{P}} \simeq T_{c_g} I_{\mathbb{P}}, \tag{1.10}$$

where $(f_g, c_g) \in \Gamma_{\mathbb{P}}$ is a generic point with S-generic $f_g \in \mathbb{P}$, and "S-generic" means the genericity in S. The isomorphism (1.10) is the reduction needed for the evasion. Intrinsically the dominance of P_l implies that the reduction (1.10) is equivalent to

$$dim\big(T_{c_g}(I_{f_g})\big) + 2 = dim\big(T_{c_g} I_{\mathbb{P}}\big). \tag{1.11}$$

Since the $GL(2)$ group has dimension 4, the minimum dimension of $T_{c_g} I_{\mathbb{P}}$ must be 6 in which case,

$$dim\big(T_{c_g}(I_{f_g})\big) = 4. \tag{1.12}$$

or equivalently $\mathbf{d}\nu_0|_{c_g}$ is surjective. Therefore the realization of the minimum dimension 6 is equivalent to the main statement of Theorem 1.1.

[3] The migration from algebraic geometry to differential geometry is our principle idea. However, the original problem in algebraic geometry stays the same. For instance, there were moduli formulations leading to the matrix representation of the surjectivity. See p 295, [3] or Lemma 1.24, [2].

1.2.2 Differential calculation

The main purpose of the reduction is to evade the genericity of the quintic 3-fold. This is achieved by switching the focus in the differentiation from the evaluation point of the partial derivatives to the tangential directions of the partial derivatives. To see it, we complete the setting. Let $\mathbb{P}^\circ = S^\circ \cap \mathbb{P}$ be the affine plane (in a projective space) spanned by three quintics f_0, f_1, f_2 (in a linear space). Similarly we continue the evaluation (1.6) for $t_i \in \mathbb{A}$. Bézout's theorem asserts

$$\Gamma_{\mathbb{P}^\circ} = \Gamma_{\mathbb{P}} \cap (\mathbb{P}^\circ \times M_d)$$

is the zero locus of the $5d+1$ coordinates' components of (1.7), i.e.

$$f(c(t_1)) = \cdots = f(c(t_{5d+1})) = 0 \tag{1.13}$$

for the varied $f \in \mathbb{P}^\circ$ (the setting requires the affine open set $\mathbb{P}^\circ$ and $\mathbb{A}$). The projection $P_r(\Gamma_{\mathbb{P}^\circ})$ is therefore the scheme defined by the resultants, i.e the ideal of the projection scheme is generated by polynomials (in c),

$$\begin{vmatrix} f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \\ f_2(c(t_j)) & f_1(c(t_j)) & f_0(c(t_j)) \\ f_2(c(t_k)) & f_1(c(t_k)) & f_0(c(t_k)) \end{vmatrix} \tag{1.14}$$

for $1 \leq i, j, k \leq 5d+1$ (the determinants are due to the linearity of (1.13) in the variable f). Assume $\mathbf{t}$ is generic in $\bigoplus_{5d+1} \mathbb{A}$. Then these equations can be localized to only $5d-1$ equations at a generic point due to the reduction (1.10) which requires the following condition (satisfied by the generic $\mathbb{P}$ above),

Pencil condition 1.3. *(for $\mathbb{P}$): For a generic $c_g \in I_{\mathbb{P}}$ and two quintics f_g, f_e in $\mathbb{P}$ with a generic $f_g \in \mathbb{P}$, $div(f_g) \cap div(f_e)$ does not contain c_g.*

Pencil condition is a 1st order condition and will be proved in Proposition 2.5. So we continue to show the localization by assuming the pencil condition. Let $c_g \in I_{\mathbb{P}^\circ}$ be generic. If for the three quintics f_0, f_1, f_2 in (1.14), the subspace

$$\Lambda_{c_g} = span\left\{ \Big(f_2(c_g(t)), f_1(c_g(t)), f_0(c_g(t)) \Big) \right\}_{t \in \mathbb{A}}$$

in $\mathbb{C}^3$ had dimension one. Then there would've been two linearly independent vectors β_1, β_2 in $\mathbb{C}^3$ such that

$$\beta_i \cdot \Lambda_{c_g} = 0, i = 1, 2,$$

where $\cdot$ is the "dot" product. Thus two quintic 3-folds

$$f_g = \beta_1 \cdot (f_2, f_1, f_0), \quad \textit{and} \quad f_e = \beta_2 \cdot (f_2, f_1, f_0)$$

would've contained c_g. Notice one of them in $span(f_g, f_e)$ must be generic in $\mathbb{P}$(because $c_g \in I_{\mathbb{P}}$ is generic). This is a violation of the pencil condition. So

$dim(\Lambda_{c_g}) \geq 2$. Hence $dim(\Lambda_c) \geq 2$ for all c in a neighborhood of c_g. Thus we obtain two linearly independent 3-dimensional vectors

$$\begin{pmatrix} f_2(c(t_1)), f_1(c(t_1)), f_0(c(t_1)) \end{pmatrix}$$
$$\begin{pmatrix} f_2(c(t_2)), f_1(c(t_2)), f_0(c(t_2)) \end{pmatrix}$$

for each c in a neighborhood of c_g, where $(t_1, t_2) \in \mathbb{A}^2$ is fixed but generic. These two vectors span the plane Λ'_c (depending on c) in $\mathbb{C}^3$. Then if

$$\begin{vmatrix} f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \\ f_2(c(t_1)) & f_1(c(t_1)) & f_0(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix} = 0 \tag{1.15}$$

for $i = 3, \cdots, 5d+1$ at c in the neighborhood, the first row

$$\begin{pmatrix} f_2(c(t_i)), f_1(c(t_i)), f_0(c(t_i)) \end{pmatrix}, \quad i = 1, \cdots, 5d+1$$

must lie in the same two dimensional plane Λ'_c. This implies that polynomials of (1.14) vanish at the same c. Thus if we let $U_{I_{\mathbb{P}}}$ be the restriction of $I_{\mathbb{P}^\circ}$ to the neighborhood and

$$b_i(c) = \begin{vmatrix} f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \\ f_2(c(t_1)) & f_1(c(t_1)) & f_0(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix} \tag{1.16}$$

$i = 3, \cdots, 5d+1$ be the $5d-1$ polynomials, then $U_{I_{\mathbb{P}}}$ is the scheme defined by $5d-1$ equations

$$b_i(c) = 0, i = 3, \cdots, 5d+1$$

in the neighborhood. So we obtain a holomorphic map,

$$\begin{array}{rcc} \nu_1 : M_d & \rightarrow & \mathbb{C}^{5d-1} \\ c & \rightarrow & \begin{pmatrix} b_3(c), b_4(c), \cdots, b_{5d+1}(c) \end{pmatrix}. \end{array}$$

such that $(\nu_1)^{-1}(\{0\})$ restricted to a neighborhood of c_g is $U_{I_{\mathbb{P}}}$. The localization shows

$$ker(\mathbf{d}\nu_1|_{c_g}) = T_{c_g}U_{I_{\mathbb{P}}} = T_{c_g}I_{\mathbb{P}}, \tag{1.17}$$

for generic $c_g \in I_{\mathbb{P}}$. Thus if $\mathbf{d}\nu_1$ is surjective, $T_{c_g}I_{\mathbb{P}}$ has the minimum dimension 6. The minimum dimension will be confirmed by the following theorem.

Theorem 1.4. *Let the plane $\mathbb{P} \subset S$ be generic. Then for generic $\mathbf{t}$, the differential map $\mathbf{d}\nu_1$ is surjective at a generic $c_g \in I_{\mathbb{P}}$.*

Remark

(1) The focus is shifted to the tangent vectors in Theorem 1.4 that now involve the basis quintics of $\mathbb{P}$. For instance, the genericity of c_g implies the genericity of the quintic 3-fold which is no longer the focus.

(2) Due to the differential geometric nature of the argument, Theorem 1.4 holds in a much larger category of projective varieties. This will be discussed elsewhere.

Our principle idea is the conversion to differential geometry, i.e the non-moduli view. The structural key in this view is the reduction (1.10) and Theorem 1.4 is its technical computation. The proof of Theorem 1.4 is completed by computing a specific Jacobian matrix of the differential map whose difficulty has been shifted to the tangent vectors represented by 2 basis quintics of $\mathbb{P}$. Technically, we first add 6 coordinates' components in the target space to expand ν_1 to a new holomorphic map $\nu_2 : M_d \to \mathbb{C}^{5d+5}$. The surjectivity of $\mathbf{d}\nu_2$ implies the surjectivity of $\mathbf{d}\nu_1$ at the same point. Then we use special types of analytic coordinates –polar types of analytic coordinates (built upon the 2 basis quintics of $\mathbb{P}$) to divide this particular representation, i.e. the Jacobian matrix of $\mathbf{d}\nu_2$ into 4 blocks. Each block can be computed in coordinates to finally obtain the non-degeneracy of the matrix.

The non-canonical adjustments for the plane $\mathbb{P}$ is confirmed by the bigger picture which shows the non-canonical nature of the existence for the decomposition

$$T_{c_g} M_d \simeq \mathbb{C}^{5d-1} \oplus \mathbb{C}^6. \tag{1.18}$$

In this extrinsic setting, the objects are non-moduli and dependent of extrinsic and intrinsic data which will be referred to as Jacobian data.

Definition 1.5. *(Jacobian data). We define the Jacobian data to be the collection of following choices: quintics $\{f_0, f_1, f_2\}$, a point $c_g \in I_{\mathbb{P}}$, $\mathbf{t} \in \prod_{5d+1} \mathbf{P}^1$, analytic charts of M_d, and affine open sets for the evaluation in (1.6), etc.*

The rest of paper is devoted to the detail to verify each statement above. It will be organized as follows. In section 2, we study the first order deformation. It proves that Theorem 1.1 is the first order consequence of Theorem 1.4. In particular, the pencil condition 1.3 will be proved. In section 3, we give the proof of Theorem 1.4 by constructing the specialization of matrices.

Acknowledgment Thanks are due to my wife Jessie Liu for creating the great environment.

2 First order deformation of rational curves

In this section, we show that both statements of Theorem 1.1 are the consequences of Theorem 1.4. The argument is in moduli view.

2.1 First order deformations of the pair

Due to the linearity in quintics for the incidence relation, the first order deformation of the pairs have a particular expression.

Lemma 2.1. *Let*

$$\begin{array}{ccc} & \Gamma & \\ {}^{P_l'}\swarrow & & \searrow^{P_r'} \\ S & & M_d \end{array} \tag{2.1}$$

where Γ *is the subscheme of*

$$\{(f,c) \in S \times M_d, C \subset div(f)\}$$

with projections P_l', P_r' *such that* P_l' *restricted to each component dominants* S*. Then if* Γ *is non-empty, at a point* $(f,c) \in \Gamma$ *with the generic* $f \in S$*,*

$$\mathbf{d}P_l' : T_{(f,c)}\Gamma \quad \rightarrow \quad T_f S \tag{2.2}$$

is surjective.

Proof. The dominance implies P_l' is smooth onto a Zariski open set of S. Thus the differential is surjective. $\square$

Next we express the pointwise derivatives in a differential sheaf.

Let

$$\begin{array}{ccc} \pi : M_d \times \mathbf{P}^1 & \rightarrow & \mathbf{P}^n, \quad n = 3,4 \\ (c,t) & \rightarrow & [c(t)]. \end{array} \tag{2.3}$$

be the morphism. (Note: The evaluation (1.4) is an affine expression of π.). Its differential map induces a homomorphism

$$\begin{array}{ccc} \pi_s : T_{c_g} M_d & \rightarrow & H^0(c_g^*(T_{\mathbf{P}^n})) \\ \alpha & \rightarrow & \pi_s(\alpha), \end{array} \tag{2.4}$$

where $\pi_s(\alpha)$ is the section whose restriction to each $t \in \mathbf{P}^1$ is $\mathbf{d}\pi(\alpha, 0)$ at the point $\pi(c_g, t) \in \mathbf{P}^n$. On the other hand, the map $(f, c) \to c^*(f)$, in non-moduli view, is a holomorphic map

$$F : S^\circ \times M_d \quad \to \quad H^0\bigg(\mathscr{O}_{\mathbf{P}^1}(5d)\bigg) \tag{2.5}$$

with $F^{-1}(\{0\}) = \Gamma \cap (S^\circ \times M_d)$.

Let $c_g \in M_d$ be non-zero, $\alpha \in T_{c_g} M_d$ and $v = \pi_s(\alpha)$. Let $f \in S^\circ$. For each $t \in \mathbb{A} \subset \mathbf{P}^1$, the pointwise derivative (a complex number),

$$\begin{array}{ccc} \mathbf{P}^1 & \to & \mathbb{C} \\ t & \to & \mathbf{d}\pi(\alpha, 0)(f)|_{c_g(t)} \end{array}$$

generates the stalk of a presheaf of module on $\mathbf{P}^1$. Such a presheaf is denoted by

$$\frac{df}{dv}. \tag{2.6}$$

Note: The $\frac{df}{dv}$ is generated by the pointwise derivative that measures whether the first order deformation v of the rational curve is inside of the quintic f. It is independent of choices of affine open sets.

Lemma 2.2. *The presheaf $\frac{df}{dv}$ is a sheaf of module generated by the global section*

$$(0, \alpha)(F)|_{(f, c_g)}$$

where $(0, \alpha) \in T_{(f, c_g)}(S^\circ \times M_d)$, and $\pi_s(\alpha) = v$.

Proof. By the chain rule, $\frac{df}{dv}$ is a sheaf. Let $t_0 \in \mathbf{P}^1$ be a point. Let U be a neighborhood of $c_g(t_0)$ and $c_g^{-1}(U)$ be neighborhood of t_0. Then the sheaf in the open set $c_g^{-1}(U)$ is generated by the directional derivative $\mathbf{d}\pi(\alpha, 0)(f)|_{c_g(t)}, t \in c_g^{-1}(U)$. By the definition the directional derivative is another derivative $(0, \alpha)(F)|_{(f, c_g)}$ in the open set $c_g^{-1}(U)$. □

Definition 2.3. *Lemma 2.1 implies the existence of the first order deformation of rational curves, that can be expressed in two ways. For that, we define two expressions: one is the superscript v^f, the other is the subscript v_f.*

1) Superscript. Let $(f_g, c_g) \in \Gamma$ be a point such that f_g is S-generic. Let $f \in S^\circ$ be another quintic. Now we work in the affine open set S° with $f_g \in S^\circ$. We denote the vector $\in T_{f_g} S^\circ$ from f_g to f by $\overrightarrow{f}$. Then Lemma 2.1 implies that there is a vector $v^f \in T_{c_g} M_d$ (with the superscript) such that

$$(\overrightarrow{f}, -v^f) \in T_{(f_g, c_g)}\Gamma, \tag{2.7}$$

where the correspondence $f \to v^f$ is a linear map unique modulo $ker(\mathbf{d}P'_r)$. It is equivalent to the existence of v^f such that

$$(0, v^f)(F)|_{(f_g, c_g)} = (\overrightarrow{f}, 0)(F)|_{(f_g, c_g)}.$$

2) Subscript. We'll fix the quintic f_g and denote $\pi_s(v^f)$ by

$$v_f.$$

(with the subscript).

Lemma 2.4. *With the notations above, the sheaf*

$$\frac{\not{d} f_g}{\not{d} v_f}$$

is generated by the global section $c_g^*(f)$.

Proof. By Lemma 2.2, the sheaf $\frac{\not{d} f_g}{\not{d} v_f}$ is generated by the section

$$(0, v^f)(F)|_{(f_g, c_g)}$$

which by Definition 2.3 is

$$(\overrightarrow{f}, 0)(F)|_{(f_g, c_g)}.$$

Notice F is a linear function in f. Thus

$$(\overrightarrow{f}, 0)(F)|_{(f_g, c_g)} = c_g^*(f).$$

□

Existence of Γ is proved in [4]. Theorem 1.1 is the further statement on its property. So we assume the dominance of P'_l, and use the notations from Lemma 2.1 to Lemma 2.4 throughout the paper.

2.2 Pencil condition

Lemma 2.1 asserts each 1-dimensional deformation of a generic quintic carries a 1-dimensional deformation of the rational curve. Pencil condition is a further description that asserts each 2-dimensional deformation of a generic quintic carries a 2-dimensional deformation of the rational curve.

Proposition 2.5. *The pencil condition holds for a generic plane $\mathbb{P}$.*

Proof. Let $c_g \in I_{\mathbb{P}}$ be generic. Let $f_g \in \mathbb{P}$ be a quintic such that $C_g \subset div(f_g)$. By the genericity of $\mathbb{P}$ in the Grassmannian, the other quintic f_e can't contain C_g.

□

In deformation, the pencil condition is a first order condition further than Lemme 2.1. This is the condition needed to reduce the problem to the surjectivity of the differential $\mathbf{d}\nu_1$.

2.3 Zariski tangent spaces

We convert the tangential property of the moduli of rational curves to the tangential property for rational curves on 3-folds.

Proposition 2.6. *Let $(f_g, c_g) \in \Gamma$ and f_g be S-generic. Then*

(a)

$$\frac{T_{c_g} I_{f_g}}{ker(\pi_s)} \simeq H^0(c_g^*(T_{X_g})). \tag{2.8}$$

where $ker(\pi_s)$ is the line in $T_{c_g} I_{f_g}$ and $X_g = div(f_g)$.

(b) If $dim(T_{c_g} I_{f_g})$=4, then

(1) c_g is an immersion,

(2) and

$$H^0(N_{c_g/X}) = 0. \tag{2.9}$$

Remark So the part (b) reduces Theorem 1.1 to the assertion of Theorem 1.4, i.e. $dim(T_{c_g} I_{f_g})$=4.

Proof. (a). Recall in (2.4)

$$\begin{array}{rcl} \pi_s : T_{c_g} M_d & \to & H^0(c_g^*(T_{\mathbf{P}^4})) \\ \alpha & \to & \pi_s(\alpha). \end{array} \tag{2.10}$$

Let's analyze it. Let $M^0, \cdots, M^4$ be the subsets of $T_{c_g} M_d$ that are the 5-tuple of $H^0(\mathscr{O}_{\mathbf{P}^1}(d))$ in M, i.e.

$$M^0 \oplus \cdots \oplus M^4 = T_{c_g} M_d.$$

Because c_g is birational to image, through the rational projections of $\mathbf{P}^4$ to its five coordinates planes of z_0, z_1, z_2, z_3, z_4, we obtain the 5 identity maps

$$M^i \quad \to \quad H^0(c_g^*(\mathscr{O}_{\mathbf{P}^4}(1)))$$

for $i = 0, \cdots, 4$. Then the direct sum gives an isomorphism

$$\pi_s|_{M^0} \oplus \cdots \oplus \pi_s|_{M^5} :$$

$$M^0 \oplus \cdots \oplus M^4 \quad \stackrel{\simeq}{\to} \quad H^0\left(\bigoplus_5 c_g^*\big(\mathscr{O}_{\mathbf{P}^4}(1)\big)\right). \tag{2.11}$$

Projectivizing both sides, we obtain that π_s is surjective and has one dimensional kernel. Then Lemma 2.2 asserts for each $\alpha \in T_{c_g} M_d$, $\pi_s(\alpha)$ lies in $H^0(c_g^*(T_{X_g}))$ if and only if $(0, \alpha)(F)|_{(f_g, c_g)} = 0$ which is equivalent to $\alpha \in T_{c_g} I_{f_g}$, i.e. the restriction map

$$\pi_s|_{T_{c_g} I_{f_g}} : T_{c_g} I_{f_g} \quad \to \quad H^0(c_g^*(T_{X_g}))$$

is also surjective. Notice $ker(\pi_s)$ is one dimensional and contained in $T_{c_g} I_{f_g}$. We complete the proof of part (a).

(b) If $dim(T_{c_g} I_{f_g})$=4, then by part (a)

$$dim(H^0(c_g^*(T_{X_g}))) = 3. \tag{2.12}$$

Now we consider it from a different point of view. Because c_g is a birational map to its image, there are finitely many points $t_i \in \mathbf{P}^1$ where the differential map

$$\mathbf{d}c_g : T_{t_i}\mathbf{P}^1 \quad \to \quad T_{c_g(t_i)}\mathbf{P}^4 \tag{2.13}$$

is a zero map. Assume its vanishing order at t_i is m_i . Let

$$m = \sum_i m_i. \tag{2.14}$$

Let $s(t) \in H^0(\mathscr{O}_{\mathbf{P}^1}(m))$ such that

$$div(s(t)) = \Sigma_i m_i t_i.$$

The sheaf homomorphism $\mathbf{d}c_g$ is injective and induces a composed bundle homomorphism ξ_s

$$T_{\mathbf{P}^1} \quad \stackrel{\mathbf{d}c_g}{\rightarrow} \quad c_g^*(T_{X_g}) \quad \stackrel{\frac{1}{s(t)}}{\rightarrow} \quad c_g^*(T_{X_g}) \otimes \mathscr{O}_{\mathbf{P}^1}(-m), \tag{2.15}$$

The induced bundle homomorphism ξ_s is injective at each point t. Let

$$N_m = \frac{c_g^*(T_{X_g}) \otimes \mathscr{O}_{\mathbf{P}^1}(-m)}{\xi_s(T_{\mathbf{P}^1})},$$

where $\xi_s(T_{\mathbf{P}^1}) \simeq T_{\mathbf{P}^1}$. So we have the exact sequence

$$0 \quad \rightarrow \quad T_{\mathbf{P}^1} \quad \stackrel{\xi_s}{\rightarrow} \quad c_g^*(T_{X_g}) \otimes \mathscr{O}_{\mathbf{P}^1}(-m) \quad \rightarrow \quad N_m \quad \rightarrow \quad 0. \tag{2.16}$$

Then

$$dim(H^0(N_m)) = dim(H^0(c_g^*(T_{X_g}) \otimes \mathscr{O}_{\mathbf{P}^1}(-m))) - 3. \tag{2.17}$$

On the other hand, three dimensional automorphism group of $\mathbf{P}^1$ gives a rise to a 3-dimensional subspace B of

$$H^0(c_g^*(T_{X_g})).$$

By the assumption in (2.12), $B = H^0(c_g^*(T_{X_g}))$. Over each point $t \in \mathbf{P}^1$, B spans one dimensional subspace. Hence

$$c_g^*(T_{X_g}) \simeq \mathscr{O}_{\mathbf{P}^1}(2) \oplus \mathscr{O}_{\mathbf{P}^1}(-k_1) \oplus \mathscr{O}_{\mathbf{P}^1}(-k_2), \tag{2.18}$$

where k_1, k_2 are some positive integers. This implies that

$$dim(H^0(c_g^*(T_{X_g}) \otimes \mathscr{O}_{\mathbf{P}^1}(-m))) = dim(H^0(\mathscr{O}_{\mathbf{P}^1}(2-m)). \tag{2.19}$$

Then

$$dim(H^0(c_g^*(T_{X_g}) \otimes \mathscr{O}_{\mathbf{P}^1}(-m))) = 3 - m. \tag{2.20}$$

Since $dim(H^0(N_m)) \geq 0$, by the formula (2.17), $m = 0$. So singular t_i does not exist. Hence c_g is an immersion.

Next we prove (2). By the exact sequence for the immersion

$$0 \quad \rightarrow \quad T_{\mathbf{P}^1} \quad \rightarrow \quad c_g^*(T_{X_g}) \quad \rightarrow \quad N_{c_g/X_g} \quad \rightarrow \quad 0.$$

we have

$$H^0(c_g^*(T_{X_g})) \simeq H^0(T_{\mathbf{P}^1}) \oplus H^0(N_{c_g/X_g}). \tag{2.21}$$

By (2.18), $H^0(N_{c_g/X_g}) = 0$.

□

Proposition 2.7. *If the pencil condition holds, then (1.10) holds, i.e*

$$T_{(f_0,c_0)}\Gamma_{\mathbb{P}} \simeq T_{c_0} I_{\mathbb{P}}, \tag{2.22}$$

where $(f_0, c_0) \in \Gamma_{\mathbb{P}}$ *is a generic point.*

Proof. By the definition of $I_{\mathbb{P}}$, the differential map $\mathbf{d}P_r$ is onto. It suffices to prove the injectivity. Let $\mathbb{P}$ be spanned by three quintics f_0, f_1, f_2, where $(f_0, c_0) \in \Gamma_{\mathbb{P}}$ is generic. Suppose $\mathbf{d}P_r$ is not injective. Then with the notation in Definition 2.3, there is a quintic $f \in \mathbb{P}$ different from f_0 (in S°) such that $(\overrightarrow{f}, 0) \in T_{(f_0,c_0)}\Gamma_{\mathbb{P}}$. Then

$$(\overrightarrow{f}, 0)(F)|_{(f_g,c_g)} = 0$$

which by Lemma 2.4 is

$$c_0^*(f) = 0. \tag{2.23}$$

The formula (2.23) indicates every point on the line through two points f_0 and f is a quintic 3-fold containing the rational curve C_0. This is a violation of the pencil condition. □

Proposition 2.7 is the last reduction necessary to prove that Theorem 1.1 is the consequence of Theorem 1.4

3 Projection of the incidence scheme

In this section, we use the Euclidean topology, i.e. the topology of complex manifolds. The topic of this section is the surjectivity of $\mathbf{d}\nu_1$, i.e. Theorem 1.4. It is divided into 4 steps. Each subsection contains one.

Subsection 3.1: In order to have a square Jacobian matrix, we add 6 coordinates' components to the original ν_1 to obtain another holomorphic map

$$\nu_2 : M_d \quad \rightarrow \quad \mathbb{C}^{5d+5}. \tag{3.1}$$

The surjectivity of $\mathbf{d}\nu_2$ at a point on $I_{\mathbb{P}}$ implies the surjectivity of $\mathbf{d}\nu_1$ at the same point.

Subsection 3.2: Let $c_g \in M_d$ be a point. We'll construct two polar types of local analytic coordinates around c_g. They will be used to analyze the matrix representation (Jacobian) of the differential map $\mathbf{d}\nu_2$.

Subsection 3.3: Specialize the Jacobian data, especially choose special quintics f_1, f_2 to adjust the the expression of the Jacobian matrix $\mathscr{A}$ of the differential $\mathbf{d}\nu_2$ in the polar types of coordinates. Then break it into block matrices to compute the blocks one-by-one.

Subsection 3.4: The subsection 3.3 is only valid around the generic point $c_g \in I_{\mathbb{P}}$. So we use $GL(2)$ action to transfer generic rational curves to all rational curves in I_f.

3.1 Holomorphic maps

In the subsection we show the the surjectivity of $\mathbf{d}\nu_1$ is induced from the surjectivity of $\mathbf{d}\nu_2$. Due to computation nature, we'll use evaluation notation (1.6) in the rest of the paper with the necessary condition that quintics lie in S° and points of $\mathbf{P}^1$ lie in $\mathbb{A}$.

Recall the definition of ν_1. First let $\mathbb{P}$ be a plane in S spanned by three arbitrary quintics f_0, f_1, f_2 in S°. Choose $5d+1$ distinct, ordered points t_i on $\mathbb{A} \subset \mathbf{P}^1$, denoted by $\mathbf{t} = (t_1, \cdots, t_{5d+1})$. Then ν_1 is just the holomorphic map

$$\begin{aligned} \nu_1 : M_d &\rightarrow \mathbb{C}^{5d-1} \\ c &\rightarrow \Big(b_3(c), b_4(c), \cdots, b_{5d+1}(c)\Big) \end{aligned} \tag{3.2}$$

where

$$b_i(c) = \begin{vmatrix} f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \\ f_2(c(t_1)) & f_1(c(t_1)) & f_0(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix}.$$

Expand the determinant $b_i(c)$ along the first row

$$\begin{vmatrix} f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \\ f_2(c(t_1)) & f_1(c(t_1)) & f_0(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix}$$
$$\|$$
$$\begin{vmatrix} f_1(c(t_1)) & f_0(c(t_1)) \\ f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix} f_2(c(t_i)) + \begin{vmatrix} f_0(c(t_1)) & f_2(c(t_1)) \\ f_0(c(t_2)) & f_2(c(t_2)) \end{vmatrix} f_1(c(t_i)) + \begin{vmatrix} f_2(c(t_1)) & f_1(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) \end{vmatrix} f_0(c(t_i))$$

for $i = 3, \cdots, 5d+1$ (to avoid the confusion, the indexes $3, \cdots, 5d+1$ must be distinguished from $1, 2$). Since the target space is the affine space $\mathbb{C}^{5d-1}$, we can express the differential map as

$$\mathbf{d}\nu_1 = \Big(\phi_3(c), \cdots, \phi_{5d+1}(c)\Big)$$

where

$$\begin{aligned} \phi_i(c) = &\begin{vmatrix} f_1(c(t_1)) & f_0(c(t_1)) \\ f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix} \mathbf{d}f_2(c(t_i)) + \begin{vmatrix} f_0(c(t_1)) & f_2(c(t_1)) \\ f_0(c(t_2)) & f_2(c(t_2)) \end{vmatrix} \mathbf{d}f_1(c(t_i)) \\ &+ \begin{vmatrix} f_2(c(t_1)) & f_1(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) \end{vmatrix} \mathbf{d}f_0(c(t_i)) + \textstyle\sum_{l=0,j=1}^{l=2,j=2} h^i_{lj}(c)\mathbf{d}f_l(c(t_j)) \end{aligned} \tag{3.3}$$

for $i = 3, \cdots, 5d+1$, $\mathbf{d}$ is the holomorphic differential [4] on the variable c and $h^i_{lj}(c)$ are polynomial functions in c. Define three numbers at a fixed rational curve $c_g \in M_d$,

$$\delta_1 = \begin{vmatrix} f_0(c_g(t_1)) & f_2(c_g(t_1)) \\ f_0(c_g(t_2)) & f_2(c_g(t_2)) \end{vmatrix},$$

$$\delta_2 = \begin{vmatrix} f_1(c_g(t_1)) & f_0(c_g(t_1)) \\ f_1(c_g(t_2)) & f_0(c_g(t_2)) \end{vmatrix} \tag{3.4}$$

$$\delta_0 = \begin{vmatrix} f_2(c_g(t_1)) & f_1(c_g(t_1)) \\ f_2(c_g(t_2)) & f_1(c_g(t_2)) \end{vmatrix}$$

Then define the quintic 3-fold $\tilde{f}_3$ by

$$\tilde{f}_3 = \delta_2 f_2 + \delta_1 f_1 + \delta_0 f_0. \tag{3.5}$$

Then the evaluation at c_g yields

$$\begin{array}{c} \mathbf{d}\nu_1|_{c_g} \\ \| \\ \left(\mathbf{d}\tilde{f}_3(c(t_3)) + \sum_{l=0,j=1}^{l=2,j=2} h^3_{lj}(c_g)\mathbf{d}f_l(c(t_j)), \cdots, \mathbf{d}\tilde{f}_3(c(t_{5d+1})) + \sum_{l=0,j=1}^{l=2,j=2} h^{5d+1}_{lj}(c_g)\mathbf{d}f_l(c(t_j))\right)\Big|_{c_g}. \end{array}$$

The computation above yields

Proposition 3.1. *Let ν_2 be the regular map*

$$\nu_2 : M_d \rightarrow \mathbb{C}^{5d+5} \tag{3.6}$$

given by $5d+5$ polynomials,

$$\begin{array}{c} c \\ \downarrow \\ \Big(f_0(c(t_1)), f_0(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_2(c(t_1)), f_2(c(t_2)) \\ \tilde{f}_3(c(t_3)), \tilde{f}_3(c(t_4)), \tilde{f}_3(c(t_5)), \cdots, \tilde{f}_3(c(t_{5d})), \tilde{f}_3(c(t_{5d+1}))\Big). \end{array} \tag{3.7}$$

Its natural extension to M is also denoted by ν_2. Then the surjectivity of $\mathbf{d}\nu_2$ at the point c_g implies the surjectivity of $\mathbf{d}\nu_1$ at the same point c_g.

[4] We switch the differential to the holomorphic differential, $\partial + \bar{\partial}$ due to the polar type of coordinates used later.

Proof. As above,

$$\mathbf{d}\nu_1|_{c_g} = \Big(\phi_3(c_g), \cdots, \phi_{5d+1}(c_g)\Big)$$

where

$$\phi_i(c_g) = \mathbf{d}\tilde{f}_3(c(t_i))\Big|_{c_g} + \sum_{l=0,j=1}^{l=2,j=2} h^i_{lj}(c_g)\mathbf{d}f_l(c(t_j))\Big|_{c_g}. \tag{3.8}$$

If $\mathbf{d}\nu_2$ is surjective at the point c_g, then $5d+5$ differentials,

$$\begin{array}{c} \mathbf{d}f_0(c(t_1)), \mathbf{d}f_0(c(t_2)), \mathbf{d}f_1(c(t_1)), \mathbf{d}f_1(c(t_2)), \mathbf{d}f_2(c(t_1)), \mathbf{d}f_2(c(t_2)) \\ \mathbf{d}\tilde{f}_3(c(t_3)), \mathbf{d}\tilde{f}_3(c(t_4)), \mathbf{d}\tilde{f}_3(c(t_5)), \cdots, \mathbf{d}\tilde{f}_3(c(t_{5d})), \mathbf{d}\tilde{f}_3(c(t_{5d+1})). \end{array} \tag{3.9}$$

are linearly independent in the stalk $\Omega_{M_d}|_{c_g}$. It follows from the linear algebra that the particular linear expression of formula (3.8) (in the basis (3.9)) shows that the differentials

$$\phi_3(c_g), \cdots, \phi_{5d+1}(c_g)$$

are also linearly independent in the same stalk $\Omega_{M_d}|_{c_g}$. Hence the differential map $\mathbf{d}\nu_1$ is surjective at the same point c_g.

□

Remark The content is this subsection is formal in algebra.

3.2 Polar and quasi-polar coordinates

Proposition 3.1 reduces Theorem 1.4 to the surjectivity of $\mathbf{d}\nu_2$ whose root lies in the higher order deformations of the rational curves. However, it is profoundly beyond a formal neighborhood. To attack it, we introduce the technique in transcendental geometry – polar types of coordinates whose main character is its non-algebraic nature.

We consider a particular type of resolution for the variety M:

$$\begin{array}{ccccccc} M & \stackrel{\sigma_1}{\longleftarrow} & \mathbb{C}^5 \times \mathbb{A}^{5d} & \stackrel{\sigma_2}{\longrightarrow} & \mathbb{C}^5 \times \mathbb{A}^{3d} \times sym^{2d}(\mathbf{P}^1) & \stackrel{\sigma_3}{\longleftarrow} & \mathbb{C}^5 \times \mathbb{A}^{3d} \times \mathbb{A}^{2d} \\ \nu_2 \downarrow & & & & & & \\ \mathbb{C}^{5d+5} & & & & & & \end{array}$$

which is described as follows. For the domain of σ_1, we denote the 5 tuples of coordinates of $\mathbb{A}^{5d}$ in the order by

$$\begin{array}{cccccc} \theta^0_1, & \theta^0_2, & \cdots, & \theta^0_d, & \Leftarrow & \text{1st tuple} \\ \vdots & \vdots & \cdots & \vdots & & \\ \theta^4_1, & \theta^4_2, & \cdots, & \theta^4_d. & \Leftarrow & \text{5th tuple} \end{array} \tag{3.10}$$

and collectively the coordinate's vector by $\boldsymbol{\theta}$. Denote the coordinates of $\mathbb{C}^5$ by $\mathbf{r} = (r_0, r_1, \cdots, r_4)$. Let σ_1 be the regular map sending $(\mathbf{r}, \boldsymbol{\theta})$ to 5 tuples in M as

$$\begin{array}{ccc} c_0(t), & \cdots, & c_4(t) \\ \| & \cdots & \| \\ r_0 \sum_{k=1}^d (t - \theta_k^0), & \cdots, & r_4 \sum_{k=1}^d (t - \theta_k^4) \end{array}$$

where t is the variable of $\mathbb{A}$. For σ_2, we rewrite the domain of σ_1 as $\mathbb{C}^5 \times \mathbb{A}^{3d} \times \mathbb{A}^{2d}$. Let q be a quadratic homogeneous polynomial in 5 variables. Let δ_1, δ_2 be two generic complex numbers. These three items define σ_2 to be the map sending $(\mathbf{r}, \boldsymbol{\theta})$ to

$$\left(\mathbf{r}, \boldsymbol{\theta}^3, div\left(\delta_1 q\big(c_0(t), \cdots, c_4(t)\big) + \delta_2 c_3(t) c_4(t)\right)\right)$$

where $\boldsymbol{\theta}^3$ denotes the variables in the first 3 tuples in (3.10), and div is the divisor of a section of $\mathscr{O}_{\mathbf{P}^1}(2d)$. Let σ_3 be the product of the identity map on

$$\mathbb{C}^5 \times \mathbb{A}^{3d}$$

and the symmetry product map

$$\mathbb{A}^{2d} \to sym^{2d}(\mathbf{P}^1).$$

The map σ_1 by definition is a dominant and generically finite-to-one map. The map σ_2 in case of $\delta_1 = 0, \delta_2 = 1$ is an identity map dominating the target. Hence for generic complex numbers δ_1, δ_2, σ_2 is dominant and generically finite-to-one. At last σ_3 is also dominant and generically finite-to-one. So once they are restricted to some analytic neighborhoods, they are complex analytic isomorphisms. In particular, we start with a point $c_g \in M$ such that zeros of all coordinate's components of c_g are distinct. Then σ_1 is unramified at c_g and $\sigma_1^{-1}(c_g)$ is a finite set. We choose c_a to be a point in $\sigma_1^{-1}(c_g)$ and c_b to be a point of another finite set $\sigma_3^{-1}(\sigma_2(c_a))$, where σ_3 is also unramified at c_b due to the genericity of δ_1, δ_2. Then due to the differential geometric inversion, there are analytic neighborhoods $U_{c_g} \subset M$, $U_a \subset \mathbb{C}^5 \times \mathbb{A}^{5d}$ and $U_b \subset \mathbb{C}^5 \times \mathbb{A}^{3d} \times \mathbb{A}^{2d}$ centered around the unramified or non branched points c_g, c_a, c_b such that restricted maps in the resolution

$$U_{c_g} \xrightarrow{\sigma_1^{-1}} U_a \xrightarrow{\sigma_4} U_b$$

are all complex analytic isomorphisms, where $\sigma_4 = \sigma_3^{-1} \circ \sigma_2$.

Definition 3.2. *(Polar and quasi-polar coordinates) We conclude that if*

(1) $c_g \in M_d$ has $5d$ distinct zeros in $c_g(\mathbb{A})$ with 5 homogeneous coordinates planes of $\mathbf{P}^4$,

(2) for $c_g = \big(c_0(t), \cdots, c_4(t)\big)$, the polynomial

$$c_0(t)c_1(t)c_2(t)\left(\delta_1 q\big(c_0(t), \cdots, c_4(t)\big) + \delta_2 c_3(t) c_4(t)\right)$$

has $5d$ *distintic zeros in* $\mathbb{A}$,

then there exist following analytic coordinates for the neighborhood U_{c_g} *of* c_g. *The variables* $(\mathbf{r}, \boldsymbol{\theta})$ *form an analytic chart* (U_{c_g}, σ_1^{-1}) *called polar coordinates at* c_g. *We denote variables of* $\mathbb{C}^5 \times \mathbb{A}^{3d} \times \mathbb{A}^{2d}$ *by*

$$\mathbf{R} = (R_0, \cdots, R_4)$$
$$\boldsymbol{\omega} = (\omega_1, \cdots, \omega_{5d}),$$

where

$$\mathbb{A}^{3d} = \{(\omega_1, \cdots, \omega_{3d})\}, \mathbb{A}^{2d} = \{(\omega_{3d+1}, \cdots, \omega_{5d})\}.$$

These variables form analytic chart $(U_{c_g}, \sigma_4 \circ \sigma_1^{-1})$ *called quasi-polar coordinates at* c_g, *denoted by* Q_M. *We call* $\mathbf{R}, \mathbf{r}$ *the radii and* $\boldsymbol{\theta}, \boldsymbol{\omega}$ *the angles.*

Remark The analytic neighborhood U_{c_g} is thus equipped with two analytic charts: polar and quasi-polar. However, the existence has requirements and is not canonical.

Let $z_0, \cdots, z_4$ be the homogeneous variables of $\mathbf{P}^4$. With generic q, δ_1, δ_2, we let

$$f_3 = z_0 z_1 z_2 (\delta_1 q + \delta_2 z_3 z_4). \tag{3.11}$$

be a quintic polynomial. Let $c_g \in M_d$ be any point such that

$$f_3(c_g(t))$$

is not a zero polynomial in $t \in \mathbb{A}$. For the same data q, δ_1, δ_2, we also assume the the associated quasi-polar coordinates Q_M exist. Denote $5d$ distinct zeros of $f_3(c_g(t))$ by

$$\mathring{t}_1, \mathring{t}_2, \cdots, \mathring{t}_{5d}$$

and all lie in $\mathbb{A}$. Then $f_3(c(\mathring{t}_i))$ for a fixed i is regarded as a holomorphic function on the Q_M neighborhood in M_d.

Proposition 3.3. *Then the Jacobian matrices for the set of polynomial functions* $f_3(c(\mathring{t}_i))$ *at the point* c_g *have simple representations in quasi-polar coordinates as follows. Let*

$$\begin{array}{cc} b_1 = & \dfrac{\partial f_3(c_g(\mathring{t}_1))}{\partial \omega_1} \\ \vdots & \vdots \\ b_{5d} = & \dfrac{\partial f_3(c_g(\mathring{t}_{5d}))}{\partial \omega_{5d}}. \end{array} \tag{3.12}$$

Then they are non-zeros and the Jacobian matrix evaluated at c_g *is diagonal:*

$$\frac{\partial(f_3(c_g(\mathring{t}_1)), \cdots, f_3(c_g(\mathring{t}_{5d}))}{\partial(\omega_1, \cdots, \omega_{5d}, R_0, \cdots, R_4)} = \begin{pmatrix} b_1 & 0 & \dots & 0 & 0\cdots & 0 \\ 0 & b_2 & \dots & 0 & 0\cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_{5d} & 0\cdots & 0 \end{pmatrix}. \tag{3.13}$$

Proof. Note $\omega_i, i = 1, \cdots, 5d$ are distinct in $\mathbb{A}$. Thus the quasi coordinates in Definition 3.2 exist. Applying Q_M coordinates to the local holomorphic functions in the form $f_3(c(t))$, we have

$$f_3(c(t)) = r_0 r_1 r_2 R_0 \prod_{i=1}^{5d} (t - \omega_i). \tag{3.14}$$

in the neighborhood U_{c_g}, where R_0 is the analytic function

$$\delta_1 q(r_0, \cdots, r_4) + \delta_2 r_3 r_4$$

of the radii. Proposition 3.3 follow from the partial derivatives of the expression (3.14). We complete the proof. □

Remark The content of this subsection only holds in transcendental geometry due to the inverse function theorem.

3.3 The specialization and deformation

The proof of Theorem 1.4: By Proposition 3.1, it amounts to show the surjectivity of $\mathbf{d}\nu_2$ at a point. Notice the surjectivity is an open condition. Our idea is to select a specific Jacobian data not only to have the intrinsic surjectivity but also the extrinsic accessibility. In particular, it includes the polar types of coordinates. In the following we divide them into 3 types: quintics f_0, f_1, f_2, rational curve c_g, and $5d$ points $t_i \in \mathbb{A}$.

Let $z_0, \cdots, z_4$ be the homogeneous coordinates of $\mathbf{P}^4$. Let f_0 be S-generic. Let

$$f_2 = z_0 z_1 z_2 z_3 z_4,$$
$$f_1 = z_0 z_1 z_2 q,$$

where q is a generic quadratic homogeneous polynomial in $z_0, \cdots, z_4$. The affine set S° is the collection of all quintics with non-zero $z_0 z_1 z_2 z_3 z_4$ term.

Let

$$c_g \in I_{\mathbb{P}}$$

be a point that has coordinate's components,

$$c_g = (c_0, \cdots, c_4).$$

By choosing a generic homogeneous coordinate's system, we may assume c_g has $5d$ distinct intersections in $c_g(\mathbb{A})$ with coordinate's planes, i.e. the five coordinate's components

$$c_i(t), i = 0, \cdots, 4$$

have $5d$ distinct zeros in $\mathbb{A}$. In the following we choose $5d+1$ distinct points t_i on $\mathbb{A} \subset \mathbf{P}^1$, denoted by $\mathbf{t} = (t_1, \cdots, t_{5d+1})$.

(1) Let t_{5d+1}, t_1, t_2 be generic and variables t_1, t_2, z_i, q, c_g satisfy

$$\begin{vmatrix} f_2(c_g(t_1)) & f_1(c_g(t_1)) \\ f_2(c_g(t_2)) & f_1(c_g(t_2)) \end{vmatrix} = 0. \tag{3.15}$$

(2) Let

$$\begin{aligned} \delta_1 &= \begin{vmatrix} f_0(c_g(t_1)) & f_2(c_g(t_1)) \\ f_0(c_g(t_2)) & f_2(c_g(t_2)) \end{vmatrix}, \\ \delta_2 &= \begin{vmatrix} f_1(c_g(t_1)) & f_0(c_g(t_1)) \\ f_1(c_g(t_2)) & f_0(c_g(t_2)) \end{vmatrix}. \end{aligned} \tag{3.16}$$

Then the ratio of δ_1, δ_2 is generic in $\mathbb{C}$ because f_0 is a generic[5]. Notice the polynomial in (3.11) evaluated at c_g is

$$f_3(c_g(t)) = c_0(t)c_1(t)c_2(t)\bigg(\delta_1 q\big((c_g(t))\big) + \delta_2 c_3(t)c_4(t)\bigg). \tag{3.17}$$

Therefore the first two zeros of the coordinate's component $c_0(t) = 0$, $\mathring{\theta}_1^0, \mathring{\theta}_2^0$ are zeros of $f_3(c_g(t)) = 0$. Let $t_3, \cdots, t_{5d}$ be the rest $5d-2$ zeros.

Now we can combine the selection with the expression of $\mathbf{d}\nu_2$. Recall in the formula (3.7) there are $5d+5$ functions. We divide them two groups: the first group with $5d-2$ functions

$$\tilde{f}_3(c(t_3)), \tilde{f}_3(c(t_4)), \cdots, \cdots, \cdots, \tilde{f}_3(c(t_{5d-2})), \tilde{f}_3(c(t_{5d-1})), \tilde{f}_3(c(t_{5d})) \tag{3.18}$$

denoted by $\mathbf{F}_1$; the second ordered group with 7 functions,

$$\tilde{f}_3(c(t_{5d+1})), f_2(c(t_1)), f_2(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_0(c(t_1)), f_0(c(t_2), \tag{3.19}$$

denoted by $\mathbf{F}_2$. With above choices, $[\delta_1, \delta_2], q$ are all generic. So we can choose the quasi-polar coordinates Q_M, defined in Definition 3.2, to be the local coordinates around $c_g \in I_{\mathbb{P}}$. We divide the Q_M coordinates also into two groups: the first ordered group with $5d-2$ variables,

$$\omega_3, \cdots, \omega_{5d} \tag{3.20}$$

denoted by $\mathbf{w}_1$; the second ordered group with 7 variables (most are radii),

$$\omega_1, \omega_2, R_0, R_1, R_2, R_3, R_4. \tag{3.21}$$

denoted by $\mathbf{w}_2$. Then the representation of the differential map $\mathbf{d}\nu_2|_{c_g}$ in Q_M coordinates can be written as

$$\mathscr{A} = \frac{\partial(\mathbf{F}_1, \mathbf{F}_2)}{\partial(\mathbf{w}_1, \mathbf{w}_2)}\bigg|_{c_g}, \tag{3.22}$$

[5]The existence of δ_1, δ_2 is the pencil condition. As in the reduction (1.10), the condition is necessary for the calculation to continue.

(where a row vector is the partial derivatives of the same function). Above divisons allow us to divide $\mathscr{A}$ to 4 blocks.

$$\left.\begin{pmatrix} \frac{\partial \mathbf{F}_1}{\partial \mathbf{w}_1} & \frac{\partial \mathbf{F}_1}{\partial \mathbf{w}_2} \\ \\ \frac{\partial \mathbf{F}_2}{\partial \mathbf{w}_1} & \frac{\partial \mathbf{F}_2}{\partial \mathbf{w}_2} \end{pmatrix}\right|_{c_g}. \tag{3.23}$$

Due to the selection (3.15), in the calculation of (3.23), we can replace $\tilde{f}$ with f. Applying Proposition 3.3, we found that $\left.\frac{\partial \mathbf{F}_1}{\partial \mathbf{w}_1}\right|_{c_g}$ is a non-zero diagonal matrix and

$$\left.\frac{\partial \mathbf{F}_1}{\partial \mathbf{w}_2}\right|_{c_g} = (0).$$

Therefore to show $\mathscr{A}$ is non-degenerate, it suffices to show

$$det(\left.\frac{\partial \mathbf{F}_2}{\partial \mathbf{w}_2}\right|_{c_g}) \neq 0, \tag{3.24}$$

where explicitly

$$\begin{array}{c} \frac{\partial \mathbf{F}_2}{\partial \mathbf{w}_2} \\ \| \\ \frac{\partial (f_3(c(t_{5d+1})), f_2(c(t_1)), f_2(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_0(c(t_1)), f_0(c(t_2)))}{\partial(\omega_1, \omega_2, R_0, R_1, R_2, R_3, R_4)}. \end{array} \tag{3.25}$$

is a 7×7 matrix. Next we adjust each variable in the following to reduce the determinant.

I) Genericity of t_{5d+1}. The genericity of q makes the following curve in $\mathbb{C}^7$,

$$\left(\frac{\partial f_3(c(t))}{\partial \omega_1}, \frac{\partial f_3(c(t))}{\partial \omega_2}, \frac{\partial f_3(c(t))}{\partial R_0}, \cdots, \frac{\partial f_3(c(t))}{\partial R_4}\right), t \in \mathbb{A} \tag{3.26}$$

span the entire space $\mathbb{C}^7$. This means the first row vector of the matrix

$$\left.\frac{\partial \mathbf{F}_2}{\partial \mathbf{w}_2}\right|_{c_g}$$

is linearly independent of other 6 row vectors if t_{5d+1} is generic. Hence it suffices for us to show the non-degeneracy of the 6×6 Jacobian matrix

$$\mathscr{B}_1 = \left.\frac{\partial \Big((f_2(c(t_1)), f_2(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_0(c(t_1)), f_0(c(t_2))) \Big)}{\partial(\omega_1, \omega_2, R_0, R_1, R_2, R_3)}\right|_{c_g}$$

(the column of partial derivatives with respect to R_4 is eliminated).

II) Genericity of q. To show $\mathscr{B}_1$ is non-degenerate, it suffices to show it is non-degenerate at a special $c_s \in I_{\mathbb{P}}$. To do that, we let $\mathbb{L}_2$ be the pencil through f_0, f_2. A component $I_{\mathbb{P}}$ contains a component $I_{\mathbb{L}_2}$. We then select c_s to be a generic point of $I_{\mathbb{L}_2}$ (c_s is generic in a lower dimensional subvariety $I_{\mathbb{L}_2}$, but may not generic in $I_{\mathbb{P}}$). Because q can vary freely to a generic position as 1st, 2nd, 5th and 6th rows stay fixed, two middle rows of the matrix $\mathscr{B}_1$,

$$\begin{array}{c}(\frac{\partial f_1(c(t_1))}{\partial \theta_1^0}, \frac{\partial f_1(c(t_1))}{\partial \theta_1^1}, \frac{\partial f_1(c(t_1))}{\partial R_0}, \cdots, \frac{\partial f_1(c(t_1))}{\partial r_3})|_{c_s} \\ (\frac{\partial f_1(c(t_2))}{\partial \theta_1^0}, \frac{\partial f_1(c(t_2))}{\partial \theta_1^1}, \frac{\partial f_1(c(t_2))}{\partial R_0}, \cdots, \frac{\partial f_1(c(t_2))}{\partial R_3})|_{c_s}\end{array} \tag{3.27}$$

must be linearly independent after the reduction by the span of 1st, 2nd, 5th and 6th rows. Then we reduce the non-degeneracy of $\mathscr{B}_1$ to the non-degeneracy of 4×4 matrix

$$\mathscr{B}_2(\delta_1) = \frac{\partial\big(f_2(c(t_1)), f_2(c(t_2)), f_0(c(t_1)), f_0(c(t_2))\big)}{\partial(\omega_1, R_0, R_1, R_2)}|_{c_s}. \tag{3.28}$$

(two row vectors (3.27) are eliminated), where the dependence of δ_1 in the differentiation is denoted. Next we change the coordinates from quasi-polar to polar with the conversion formula,

$$\begin{array}{lcl}\mathbf{d}\sigma_4 : \frac{\partial}{\partial \theta_1^0} & \Rightarrow & \frac{\partial}{\partial \omega_1} + \delta_1 \beta \\ \mathbf{d}\sigma_4 : \frac{\partial}{\partial r_0} & \Rightarrow & \frac{\partial}{\partial R_0} + \delta_1 \alpha_0 \\ \mathbf{d}\sigma_4 : \frac{\partial}{\partial r_1} & \Rightarrow & \frac{\partial}{\partial R_1} + \delta_1 \alpha_1 \\ \mathbf{d}\sigma_4 : \frac{\partial}{\partial r_2} & \Rightarrow & \frac{\partial}{\partial R_2} + \delta_1 \alpha_2\end{array}$$

where $\beta, \alpha_0, \alpha_1, \alpha_2$ are fixed vectors in $T_{c_b}(U_{c_b})$. Then we obtain

$$\mathscr{B}_2(\delta_1) = \mathscr{B}_3 + \delta_1 B.$$

where B is some matrix independent of δ_1 and

$$\mathscr{B}_3 = \frac{\partial\big(f_2(c(t_1)), f_2(c(t_2)), f_0(c(t_1)), f_0(c(t_2))\big)}{\partial(\theta_1^0, r_0, r_1, r_2)}|_{c_s} \tag{3.29}$$

is in polar coordinates and clearly independent of δ_1. Since δ_1 is generic, so it suffices to prove the non-degeneracy of

$$\mathscr{B}_3 = \frac{\partial\big(f_2(c(t_1)), f_2(c(t_2)), f_0(c(t_1)), f_0(c(t_2))\big)}{\partial(\theta_1^0, r_0, r_1, r_2)}|_{c_s}$$

in polar coordinates.

III) Genericity of f_0. The partial derivatives with respect to polar variables can be converted to a multiple of partial derivatives of quintics. Applying this direct calculation, we obtain that

$$\begin{array}{c}\mathscr{B}_3\\ \|\\ \lambda\begin{pmatrix} \frac{1}{t_1-\mathring{\theta}_1^0} & 1 & 1 & 1\\ \frac{1}{t_2-\mathring{\theta}_1^0} & 1 & 1 & 1\\ \frac{\partial f_0(c_s(t_1))}{\partial \theta_1^0} & (z_0\frac{\partial f_0}{\partial z_0})|_{c_s(t_1)} & (z_1\frac{\partial f_0}{\partial z_1})|_{c_s(t_1)} & (z_2\frac{\partial f_0}{\partial z_1})|_{c_s(t_1)}\\ \frac{\partial f_0(c_s(t_2))}{\partial \theta_1^0} & (z_0\frac{\partial f_0}{\partial z_0})|_{c_s(t_2)} & (z_1\frac{\partial f_0}{\partial z_1})|_{c_s(t_2)} & (z_2\frac{\partial f_0}{\partial z_2})|_{c_s(t_2)}\end{pmatrix},\end{array} \tag{3.30}$$

where λ is a non-zero complex number from the multiples. We further compute to have

$$\begin{array}{c}\mathscr{B}_3\\ \|\\ \lambda(\frac{1}{t_1-\mathring{\theta}_1^0}-\frac{1}{t_2-\mathring{\theta}_1^0})\begin{pmatrix} 1 & 1 & 1\\ (z_0\frac{\partial f_0}{\partial z_0})|_{c_s(t_1)} & (z_1\frac{\partial f_0}{\partial z_1})|_{c_s(t_1)} & (z_2\frac{\partial f_0}{\partial z_2})|_{c_s(t_1)}\\ (z_0\frac{\partial f_0}{\partial z_0})|_{c_s(t_2)} & (z_1\frac{\partial f_0}{\partial z_1})|_{c_s(t_2)} & (z_2\frac{\partial f_0}{\partial z_2})|_{c_s(t_2)}\end{pmatrix},\end{array} \tag{3.31}$$

where $\mathring{\theta}_1^0$ is a complex number. Since all the variables t_1, t_2, z_i, q are only required to satisfy one equation (3.15), we may assume $(t_1, t_2) \in \mathbb{C}^2$ is generic. Let's now prove the non-vanishing of

$$J(f_0, c_s) = \begin{vmatrix} 1 & 1 & 1\\ (z_0\frac{\partial f_0}{\partial z_0})|_{c_s(t_1)} & (z_1\frac{\partial f_0}{\partial z_1})|_{c_s(t_1)} & (z_2\frac{\partial f_0}{\partial z_2})|_{c_s(t_1)}\\ (z_0\frac{\partial f_0}{\partial z_0})|_{c_s(t_2)} & (z_1\frac{\partial f_0}{\partial z_1})|_{c_s(t_2)} & (z_2\frac{\partial f_0}{\partial z_2}|_{c_s(t_2)}\end{vmatrix}.$$

First we identify the quintic 3-fold containing c_s as f_s. Then we consider a different Jacobian where the quintic contains c_s, i.e.

$$J(f_s, c_s) = \begin{vmatrix} 1 & 1 & 1\\ (z_0\frac{\partial f_s}{\partial z_0})|_{c_s(t_1)} & (z_1\frac{\partial f_s}{\partial z_1})|_{c_s(t_1)} & (z_2\frac{\partial f_s}{\partial z_2})|_{c_s(t_1)}\\ (z_0\frac{\partial f_s}{\partial z_0})|_{c_s(t_2)} & (z_1\frac{\partial f_s}{\partial z_1})|_{c_s(t_2)} & (z_2\frac{\partial f_s}{\partial z_2})|_{c_s(t_2)}\end{vmatrix}.$$

We then obtain

$$J(f_s, c_s) = \begin{vmatrix} f_4(c_s(t_1)) & f_5(c_s(t_1))\\ f_4(c_s(t_2)) & f_5(c_s(t_2))\end{vmatrix},$$

where

$$\begin{array}{c} f_4 = z_0\frac{\partial f_s}{\partial z_0} - z_2\frac{\partial f_s}{\partial z_2}\\ f_5 = z_1\frac{\partial f_s}{\partial z_1} - z_2\frac{\partial f_s}{\partial z_2}\end{array}$$

are two quintic 3-folds. If $J(f_s, c_s) = 0$, then by the genericity of t_1, t_2, the two dimensional vectors vectors

$$\Big(f_4(c_s(t)), f_5((c_s(t)\Big),\ \mathit{all}\ t$$

must span a line. So there exist two complex numbers ϵ_1, ϵ_2 not all zeros such that

$$(\epsilon_1 z_0 \frac{\partial f_s}{\partial z_0} + \epsilon_2 z_1 \frac{\partial f_s}{\partial z_1} + (-\epsilon_1 - \epsilon_2) z_2 \frac{\partial f_s}{\partial z_2})|_{c_s(t)} = 0. \tag{3.32}$$

Then the vector field (with the varied $t \in \mathbf{P}^1$)

$$\eta = [0, 0, \epsilon_1 c_s^0(t), \epsilon_2 c_s^1(t), (-\epsilon_1 - \epsilon_2) c_s^2(t)]$$

is the non-zero holomorphic section of $(c_s)^*(T_{X_s})$, where $div(f_s) = X_s$ is the generic quintic 3-fold and c_s^i is the i-th coordinate's component of c_s. Let $U \subset \mathbf{P}^1$ be the open set such that $c_s : U \to c(U)$ is an isomorphism. Then $c_s^*(T_{c(U)})$ is a subbundle of $(c_s)^*(T_{X_s})|_U$. Let E be the closure of $c_s^*(T_{c(U)})$ in $(c_s)^*(T_{X_s})$. Then it is also a vector bundle. Since the pushforward of a section of $T_{\mathbf{P}^1}$ is a section of E that has $m+2$ zeros, E has degree $m+2$. Notice η has nonzero reduction $\hat{\eta}$ in the quotient bundle $\frac{(c_s)^*(T_{X_s})}{E}$. Thus it determines a line bundle $L_\eta \subset \frac{(c_s)^*(T_{X_s})}{E}$. On the other hand, $\frac{(c_s)^*(T_{X_s})}{E}$ is a rank 2 bundle of degree $-2-m$. So there is a decomposition

$$\frac{(c_s)^*(T_{X_s})}{E} \simeq \mathscr{O}_{\mathbf{P}^1}(k) \oplus \mathscr{O}_{\mathbf{P}^1}(-k-2-m) \tag{3.33}$$

where k, m integers and $m \geq 0$. Since two numbers $-k-2-m, k$ can't be non-negative simultaneously, L_η could only be either

$$\mathscr{O}_{\mathbf{P}^1}(k), \ \ or \ \mathscr{O}_{\mathbf{P}^1}(-k-2-m).$$

In either case, L_η is intrinsically determined by the bundle structure of $(c_s)^*(T_{X_s})$. However L_η which is determined by η varies with extrinsic coordinates $z_0, \cdots, z_4$. The contradiction shows

$$J(f_s, c_s) \neq 0.$$

At last we deform c_s to a generic position in $I_{\mathbb{P}}$ to obtain a generic $c_g \in I_{\mathbb{P}}$ with $J(f_s, c_g) \neq 0$. We complete the proof of Theorem 1.4 for the specific $\mathbb{P}$ spanned by the chosen quintics f_0, f_1, f_2. Since the surjectivity is an open condition, we can deform three quintics in the basis to S-generic quintics. So Theorem holds for generic $\mathbb{P}$.

Remark The proof shows Theorem 1.4 does not hold for all $\mathbb{P}$, but it holds for the most of selected $\mathbb{P}$. So the assumption of the genericity of $\mathbb{P}$ in introduction is for the convenience only.

3.4 Hilbert scheme $\mathscr{M}_d(X)$

Let's prove Theorem 1.1 for ALL rational curves c. Theorem 1.4 asserts the $T_{c'} I_{\mathbb{P}} = 6$ at generic $c' \in I_{\mathbb{P}}$. Let f be such a quintic that $(f, c') \in \Gamma_{\mathbb{P}}$. Then the genericity of c' implies that $(f, c') \in \Gamma_{\mathbb{P}}$ and $f \in S$ are also generic. By the pencil condition $dim(T_{c'}I_f) = 4$. Notice I_f is a scheme whose dimension ≥ 4. Hence it must be smooth at c' of dimension 4. Notice the $GL(2)$ orbit of c' on I_f also has dimension 4. Since I_f is irreducible, the orbit is I_f. Furthermore for all $c \in I_f$, $dim(T_c I_f) = 4$. Then Proposition 2.6 implies c is an immersed rational curve such that $H^0(N_{c/X}) = 0$. Hence the Zariski tangent space of $\mathscr{M}_d(X)$ at $[C]$,

$$Hom(\mathscr{I}_C/\mathscr{I}_C^2, \mathscr{O}_C)$$

is also 0. Since $dim(\mathscr{M}_d(X)) \geq 0$, $\mathscr{M}_d(X)$ is smooth at $[C]$ of dimension 0. Theorem 1.1 is proved.

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