A proof of Hodge conjecture

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Abstract

We have shown a method of constructing cycle classes of cohomology in [9]. Consequently, we go further in this paper to show a proof of the generalized Hodge conjecture.

Contents

1 Main theorem

The Hodge conjecture was proposed in 1950 ([5]). It predicts on a smooth projective variety over $\mathbb{C}$, the filtration of cohomology determined by Hodge's bi-grading coincides with the filtration determined by subvarieties.

Key words: Hodge conjecture, coniveau, level.
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Precisely it can be described in Grothendeick's formulation as follows ([3]). Let X be a smooth projective varieties X over $\mathbb{C}$, with dimension n. Let $H(X;\mathbb{Q})$ denote the total singular cohomology with rational coefficients and we'll add superscript to denote the degree. We denote the linear span of all sub-Hodge structures of coniveau i and degree $2i+k$ by

$$M^iH^{2i+k}(X) \subset H^{2i+k}(X;\mathbb{Q})$$

where the indexes have the range $0 \leq 2i+k \leq 2n, 0 \leq i \leq n$. On the geometric side, the coniveau filtration

$$N^iH^{2i+k}(X)$$

denotes the linear span of kernels of the linear maps

$$H^{2i+k}(X;\mathbb{Q}) \quad \to \quad H^{2i+k}(X-W;\mathbb{Q}) \tag{1.1}$$

for an algebraic set W of codimension at least i. Both $N^iH^{2i+k}(X), M^iH^{2i+k}(X)$ are subspaces over $\mathbb{Q}$. Then Grothendieck proposed

Conjecture 1.1. *For whole numbers i, k satisfying $i \leq n$ and $2i+k \leq 2n$*

$$N^iH^{2i+k}(X) = M^iH^{2i+k}(X). \tag{1.2}$$

To prove the conjecture, we'll use the equivalent index, level (instead of coniveau). So we re-organize them to form two filtration, called level filtration:

(I) Geometrically leveled filtration $\mathcal{N}_\bullet(X)$. For each $k \in \{0\} \cup \mathbb{N}$,

$$\mathcal{N}_k(X) = \sum_{i=-\infty}^{\infty} N^iH^{2i+k}(X) \tag{1.3}$$

with

$$\mathcal{N}_0(X) \subset \mathcal{N}_1(X) \subset \cdots \subset H(X;\mathbb{Q}), \tag{1.4}$$

(II) Hodge leveled filtration $\mathcal{M}_\bullet(X)$. For each $k \in \{0\} \cup \mathbb{N}$,

$$\mathcal{M}_k(X) = \sum_{i=-\infty}^{\infty} M^iH^{2i+k}(X) \tag{1.5}$$

with

$$\mathcal{M}_0(X) \subset \mathcal{M}_1(X) \subset \cdots \subset H(X;\mathbb{Q}), \tag{1.6}$$

A cycle in $\mathcal{N}_k(X)$ will be called $\mathcal{N}_k$ leveled and a cycle in $\mathcal{M}_k(X)$ will be called $\mathcal{M}_k$ leveled.

By Corollary 8.2.8, [1], $\mathcal{N}_k(X)$ is the sub-Hodge structure. Hence filtrations satisfy

$$\mathcal{N}_k(X) \subset \mathcal{M}_k(X) \quad for\ all\ k, X \tag{1.7}$$

Theorem 1.2. *(**Main theorem**) Conjecture 1.1 is correct.*

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2 Idea

2.1 Real intersection theory and Grothendieck duality

The central idea is to approach the coniveau filtration in currents for which we can use the measure-theoretical analysis. The analysis builds the technical foundation – real intersection theory ([8]) that assures that there is a special type of currents inside of $\mathscr{D}'(\mathcal{X})$ (topological dual of smooth forms with a compact support), called Lebesgue currents which include singular chains and C^∞-forms, and they form a subspace denoted by $\mathscr{L}(\mathcal{X})$. For any two Lebesgue currents T_1, T_2, we defined an extrinsic intersection current as the limit of the de Rham's regularization ([2]), denoted by

$$[T_1 \wedge T_2]. \tag{2.1}$$

The intersection is also Lebesgue, but it depends on a special type of extrinsic covering on the manifold called de Rham data. This intersection is the extension of classical intersections that include the transversal intersection on real manifolds; the proper intersection on smooth projective varieties; products on cohomology ring and Chow ring. It leads to an important idea which replaces the algebraic cycles by the classes supported on the algebraic sets. As a result, we obtain the Grothendieck duality.

Theorem 2.1. *(Grothendieck duality. See [9]) Let X be a smooth projective variety over $\mathbb{C}$ with dimension n. For whole numbers p, q, k satisfying $p + q = n - k$, the homomorphism*

$$\begin{array}{ccc} N^pH^{2p+k}(X) & \to & N^qH^{2q+k}(X) \\ \alpha & \to & \alpha \cup u^{q-p} \end{array}$$

is an isomorphism, where $u \in H^2(X;\mathbb{Z})$ is the hyperplane section class.

Remark The duality for $k = 1$ is the Lefschetz standard conjecture over the complex numbers ([4]). In this paper we show the Hodge conjecture is the natural consequence of Grothendieck duality, but in need of a delicate finesse.

2.2 The finesse – induction for the Hodge conjecture

Recall the assertion of Main theorem is

$$M^pH^{2p+k}(X) = N^pH^{2p+k}(X), \tag{2.2}$$

for the indexes as in Theorem 2.1. By Corollary 8.2.8, [1], we have the inclusion

$$N^pH^{2p+k}(X) \subset M^pH^{2p+k}H(X) \quad for\ all\ p, k. \tag{2.3}$$

Since they are finitely dimensional vector spaces over a field, it is sufficient to show that they have equal dimension. Let's focus on the dimension only. By the formula (2.3),

$$dim\Big(N^pH^{2p+k}(X)\Big) \leq dim\Big(M^pH^{2p+k}H(X)\Big). \tag{2.4}$$

Applying the intersection form, we obtain the homomorphism

$$\begin{array}{ccc} M^pH^{2p+k}H(X) \simeq \Big(M^qH^{2q+k}H(X)\Big)^* & \to & \Big(N^qH^{2q+k}H(X)\Big)^* \\ & & \wr\| \\ & & \Big(N^pH^{2p+k}H(X)\Big)^* \end{array} \tag{2.5}$$

where the vertical isomorphism is the Grothendieck duality. Therefore to show

$$dim\Big(N^pH^{2p+k}(X)\Big) \geq dim\Big(M^pH^{2p+k}H(X)\Big)$$

it is sufficient to show

Proposition 2.2. *The homomorphism induced from the intersection form,*

$$M^pH^{2p+k}(X) \quad \to \quad (N^qH^{2q+k}(X))^* \tag{2.6}$$

is injective.

In terms of intersection numbers, the proposition, by the Grothendieck duality, is equivalent to the claim

Claim 2.3. *for any non-zero cycle α of Hodge level k (i.e, $\mathcal{M}_k$ leveled) there is a cycle β of geometric level k (i.e. $\mathcal{N}_k$ leveled) such that the intersection number between them is non-zero.*

The proof heavily relies on the Grothendieck duality (i.e. Theorem 2.1) that furthermore reduces the proposition to the essential case[I] in the middle

[I] Zuker in [10] also dealt with this case, but the situation is quite different due to the difference in our fundamental views.

dimension $2p + k = n$. So for the following sketch, we focus on the middle dimension only.

In the following technical arguments we'll abuse the pushforward notation $(\bullet)_*$ for singular cycles, algebraic cycles and classes in cohomology. The classical results for these operators confirm that they preserve the levels on both filtrations provided the maps are holomorphic. Also the angle bracket $\langle * \rangle$ denotes the descend of the object $*$ to cohomology.

We denote the collection of all smooth projective varieties X with

$$H^1(X; \mathbb{Q}) \neq 0 \tag{2.7}$$

by

$$Corr_1(\mathbb{C}).$$

First we assume $p \neq 1$. Since the case of $p = 0$ is trivial, so $p \geq 2$. Next we use induction on n. When $n = 1, 2, 3$, the proof for most of cases follows from the well-known Lefschetz $(1, 1)$ theorem. So let's see the main case for $n \geq 4$. Suppose Proposition 2.2, therefore Main theorem hold for all $\mathcal{X} \in Corr_1(\mathbb{C})$ of $dim(\mathcal{X}) < n$. Let's consider $X \in Corr_1(\mathbb{C})$ with

$$dim(X) = n.$$

Let $\alpha \in M^p H^{2p+k}(X)$ such that $2p + k = n$. Let E be an elliptic curve, and $a, a' \in H^1(E; \mathbb{Q})$ be a standard basis,

$$a \cup a = 0 = a' \cup a', a \cup a' = 1.$$

Let

$$Y = X \times E \in Corr_1(\mathbb{C}).$$

Next we trace the intersection numbers occurring in X and Y. Notice that

$$\alpha \otimes a' \in M^p H^{2p+k+1}(Y; \mathbb{Q}).$$

Since the Poincaré duality is compatible with the sub-Hodge structures, there is a non-zero

$$\theta \in M^p H^{2p+k+1}(Y; \mathbb{Q})$$

such that

$$(\alpha \otimes a', \theta)_Y \neq 0. \tag{2.8}$$

Let θ be generic in $M^p H^{2p+k+1}(Y; \mathbb{Q})$. We call it "dual" of $\alpha \otimes a'$. Let

$$P : Y \quad \to \quad X \tag{2.9}$$

be the projection. Taking the integration along the fibre, we have the class

$$P_*(\theta) \in M^{p-1} H^{2p+k-1}(X) \tag{2.10}$$

in $\mathcal{M}_{k+1}$. Because $\theta \in M^p H^{2p+k+1}(Y;\mathbb{Q})$ is generic and P_* is surjective, so $P_*(\theta)$ is also generic in $M^{p-1}H^{2p+k-1}(X)$. Notice $u^r \cup H^1(X;\mathbb{Q})$ (or $u^r \cup H^0(X;\mathbb{Q})$) for a some $r \geq 0$ is contained in $M^{p-1}H^{2p+k-1}(X)$. Hence the assumption $H^1(X;\mathbb{Q}) \neq 0$ implies $P_*(\theta) \neq 0$. Notice $P_*(\theta)$ is a cycle on X and by the hard Lefschetz theorem,

$$P_*(\theta) \cup u \neq 0$$

in cohomology of X. Next we move the cycles to another variety. Let

$$X_{n-1} \in Corr_1(\mathbb{C})$$

be a smooth hyperplane section of X, and

$$i : X_{n-1} \quad \rightarrow \quad X \tag{2.11}$$

be the embedding. By Lefschetz hyperplane theorem,

$$i^*(P_*(\theta))$$

is non-zero and $\mathcal{M}_{k+1}$ leveled on the lower dimensional variety X_{n-1} (Both Hodge level and geometric level are preserved under the pullback of a homolomorphic map). Since

$$X_{n-1} \in Corr_1(\mathbb{C}),$$

by the inductive assumption, $i^*(P_*(\theta))$ becomes $\mathcal{N}_{k+1}$ leveled.

Since

$$i_* \circ i^*(P_*(\theta)) = P_*(\theta) \cup u$$

and i_* is the Gysin homomorphism,

$$P_*(\theta) \cup u$$

is also $\mathcal{N}_{k+1}$ leveled. By Grothendieck duality, since the cycle

$$P_*(\theta) \cup u \in N^p H^{2p+k},$$

$$P_*(\theta) \in N^{p-1}H^{2p+k-1}(X)$$

i.e. $P_*(\theta)$ is $\mathcal{N}_{k+1}$ leveled (In this step, we turned the cycle from Hodge leveled to geometrically leveled.). Next we need an important lemma,

Lemma 2.4. *If $P_*(\theta)$ is $\mathcal{N}_{k+1}$ leveled, θ is also $\mathcal{N}_{k+1}$ leveled , i.e.*

$$\theta \in N^p H^{2p+k+1}(Y). \tag{2.12}$$

Remark However $P_*(\theta), \theta$ are from different ambient spaces.

Lemma 2.4 will be proved by a similar induction. Afterwards have the final step which is called the "descending construction". It extracts a cycle of lower geometric level from θ that has a higher geometric level. This requires the real intersection theory.

Lemma 2.5. *There is a class $\beta \in N^q H^{2q+k}(X)$ such that*

$$(\alpha \otimes a', \theta) = (\alpha, \beta) \neq 0. \tag{2.13}$$

Both lemmas 2.4, 2.5 are based on the real intersection theory.

To finish the proof, we deal with the case where $p = 1$. In this case we already have Proposition 2.2 for $p \neq 1$ for varieties in $Corr_1(\mathbb{C})$. Thus for an arbitrary smooth projective X, we take $X \times E \in Corr_1(\mathbb{C})$ to allow the coniveau=2 to obtain a non-zero intersection number. Then use the projection to get back to the original variety X. This completes all cases for Proposition 2.2 for X. At last we notice that the formula (2.2) holds on X if and only if it holds on $X \times E$ where E is an elliptic curve. We complete the proof of Proposition 2.2.

2.3 Organization

In Section 3, we give the initial verification of the induction. In Section 4, we show the inductive step. In appendix, we complete a lemma for Section 4.

3 Surfaces and threefolds

Let X be smooth projective variety over $\mathbb{C}$. In this section we prove

Proposition 3.1.

$$\mathcal{N}_k(X) = \mathcal{M}_k(X) \tag{3.1}$$

for all X of $dim(X) \leq 3$.

Proof. When X is a curve, the proposition holds by the classical theory for curves. Next we consider two cases.

3.1 Surfaces

Assume $dim(X) = 2$.

We have

$$\mathcal{N}_0(X) = \sum_{i=0}^{2} N^i H^{2i}(X; \mathbb{Q}).$$

By the Lefschetz theorem on (1,1) classes,

$$\sum_{i=0}^{2} N^i H^{2i}(X) = \sum_{i=0}^{2} M^i H^{2i}(X) = \mathcal{M}_0(X).$$

Now we consider the level 1.

$$\mathcal{N}_1(X) = \mathcal{N}_0 \oplus N^0H^1(X) \oplus N^1H^3(X).$$

Thus

$$\mathcal{N}_1(X) = \mathcal{M}_0 \oplus H^1(X;\mathbb{Q}) \oplus N^1H^3(X).$$

By the Grothendieck duality,

$$N^1H^3(X) \simeq N^0H^1(X) = H^1(X;\mathbb{Q}) = M^0H^1(X).$$

For sub-Hodge structures,

$$M^0H^1(X) \simeq M^1H^3(X).$$

Thus because $M^0H^1(X) = H^1(X;\mathbb{Q})$,

$$N^1H^3(X) \simeq M^1H^3(X).$$

Then $\mathcal{N}_1(X), \mathcal{M}_1(X)$ have the same dimension. Therefore

$$\mathcal{N}_1(X) = \mathcal{M}_1(X).$$

The maximal level $k = 2$ is a trivial case. Now we conclude

$$\mathcal{N}_k(X) = \mathcal{M}_k(X)$$

for $dim(X) = 2$.

3.2 3-folds

Assume $dim(X) = 3$.

In this case, the levels could only be $k = 0, 1, 2$.

If $k = 0$, we need to show the perfect intersection between

$$M^1H^2(X) \times N^2H^4(X). \tag{3.2}$$

$$M^2H^4(X) \times N^1H^2(X). \tag{3.3}$$

For both, we use the Lefschetz theorem for $(1,1)$ class and the Poincaré duality.

If $k = 1$, we should prove the only case

Theorem 3.2. *On any 3-dimensional smooth projective variety X over $\mathbb{C}$,*

$$N^1H^3(X) = M^1H^3(X).$$

Proof. Deligne's lemma 8.2.8, [1] implies that

$$N^1H^3(X) \subset M^1H^3(X).$$

Thus it is sufficient to prove

$$M^1H^3(X) \subset N^1H^3(X). \tag{3.4}$$

It starts with a classical result in [6] as follows. Let $L \subset H^3(X;\mathbb{Q})$ be a sub-Hodge structure of coniveau 1. Then it is polarized, and there is a smooth projective curve C, and a Hodge cycle

$$\Psi \in Hdg^4(C \times X) \tag{3.5}$$

such that

$$\Psi_*(H^1(C;\mathbb{Q})) = L. \tag{3.6}$$

where $\Psi_*(H^1(C;\mathbb{Q}))$ is the image of the correspondence of classes $\in H^1(C;\mathbb{Q})$. Next we use the real intersection theory to represent the cohomological formula (3.6) as

$$P_*\Big[T \wedge \big((\bullet) \times X\big)\Big], \tag{3.7}$$

where T is a singular cycle representing the class Ψ and $C \times X$ is equipped with a de Rham data. Also we should note that P_* is the projection of singular cycles. By Property 1.3, II, [8] represent all classes in L. Next we adjust the position of T homotopically. Notice P_* on cohomology is a Hodge morphism, so $P_*(\Psi)$ is a Hodge class of degree 2 on the 3-fold X. By the Lefschetz (1,1) theorem $P_*(\Psi)$ is algebraic on X. So there is a singular cycle T_Ψ on $C \times X$ representing the class Ψ such that the projection in singular cycles satisfies

$$P_*(T_\Psi) = S + bW \tag{3.8}$$

where S is an algebraic cycle S (a divisor of X), and bW is an exact cycle of real dimension 4 in X. Consider another singular cycle in $C \times X$

$$T := T_\Psi - [e] \times bW \tag{3.9}$$

denoted by T, where $[e]$ is a current of evaluation at a point $e \in C$. By adjusting the singular chain W continuously, we can assume the projection of the support of T satisfies

$$P(supp(T)) = supp(P_*(T)), \tag{3.10}$$

i.e. the projection of the support is the support of the projection. (See appendix for the proof). Thus we have the projection of singular cycle

$$P_*(T) = S. \tag{3.11}$$

Let Θ be the collection of closed singular cycles on C representing the classes in $H^1(C;\mathbb{Q})$. Recall we applied the real intersection theory to establish the correspondence of currents ([8]),

$$T_{\circledast}(\Theta) \tag{3.12}$$

defined as a collection of currents in the form

$$T_{\circledast}(\alpha)$$

for $\alpha \in \Theta$. So it is a collection of currents supported on the algebraic set of

$$P_*(T) = S,$$

i.e. the currents in L are all supported on the algebraic set $|S|$. This is a criterion for coniveau filtration in terms of currents, i.e. for $\beta \in T_{\circledast}(\Theta)$, the cohomology class $\langle\beta\rangle$ of β satisfies

$$\langle\beta\rangle \in ker\bigg(H^3(X;\mathbb{Q}) \quad \to \quad H^3(X - |S|;\mathbb{Q}) \bigg). \tag{3.13}$$

This shows $L \subset N^1H^3(X)$. We complete the proof.

∎

If $k = 2$, we need to show the intersection form is perfect on

$$M^1H^4(X) \times N^0H^2(X). \tag{3.14}$$

Notice

$$M^1H^4(X) \simeq M^0H^2(X).$$

Since

$$M^0H^2(X) = N^0H^2(X)$$
$$dim(M^1H^4(X)) = dim(N^0H^2(X)).$$

We complete the proof.

∎

4 Proof

Let's prove claim 2.3, i.e. Proposition 2.2. It is divided into two cases: subsection 4.1 for non-middle dimension, subsection 4.2 for middle dimension.

4.1 Non-middle dimension

The following proof is the standard verification in cohomology (except the Grothendieck duality). Suppose $q > p$. Let $\alpha \in M^pH^{2p+k}(X)$ be a non-zero cycle. Let

$$h = q - p > 0.$$

Then by the hard Lefschetz theorem $\alpha u^h \neq 0$ in $H^{2q+k}(X;\mathbb{Q})$. Let

$$Y = X \cap V^h$$

be a smooth plane section of X and

$$i : Y \quad \hookrightarrow \quad X \tag{4.1}$$

be the inclusion map. Note Y is irreducible. Then applying Lemma 6.2, [7], we obtain that

$$\alpha u^h = i_* \circ i^*(\alpha).$$

Hence $i^*(\alpha) \neq 0$ in $H^{2p+k}(Y;\mathbb{Q})$. By Proposition 5.2, [7]

$$i^*(\alpha)$$

is also $\mathcal{M}_k$ leveled. Since $h > 0$, we can apply the inductive assumption to the variety Y to obtain a $\mathcal{N}_k$ leveled cycle β such that

$$(i^*(\alpha), \beta)_Y \neq 0. \tag{4.2}$$

Then applyig Lemma 6.2 , [7], we have

$$(\alpha, i_*(\beta))_X = (i^*(\alpha), \beta)_Y \neq 0 \tag{4.3}$$

Notice by Proposition 5.2, [7], $i_*(\beta)$ is $\mathcal{N}_k$ leveled. Thus the intersection form is perfect. Next we consider the case $q < p$. Let $h = p - q > 0$. We start with

$$\alpha \in M^p H^{2p+k}(X).$$

Using hard Lefschetz theorem there is a $\alpha_h \in H^{2q+k}(X;\mathbb{Q})$ such that

$$\alpha = \alpha_h u^h. \tag{4.4}$$

By the same argument above we obtain a $\mathcal{N}_k$ leveled cycle β in $H^{2p+k}(X;\mathbb{Q})$ such that

$$(\alpha_h, \beta)_X \neq 0. \tag{4.5}$$

Now applying Theorem 2.1, there is a $\mathcal{N}_k$ leveled cycle $\beta_h \in H^{2q+k}(X;\mathbb{Q})$ such that

$$\beta_h u^h = \beta. \tag{4.6}$$

Then (4.5) becomes

$$(\alpha_h, \beta_h u^h)_X = (\alpha_h u^h, \beta_h)_X = (\alpha, \beta_h)_X \neq 0. \tag{4.7}$$

where β_h is $\mathcal{N}_k$ leveled. Thus we complete the proof of Claim 2.3 for the case $p \neq q$.

4.2 Middle dimension

Proof. of Claim 2.3 for the middle dimension, i.e. $2p + k = n$. We use an induction. Suppose Proposition 2.2, therefore Main theorem hold for all

$$\mathcal{X} \in Corr_1(\mathbb{C}), \quad 4 \le dim(\mathcal{X}) \le n-1.$$

We assume $X \in Corr_1(\mathbb{C})$ with $dim(X) = n$. We would like to prove the claim 2.3, i.e. for $\alpha \in M^p H^{2p+k}(X; \mathbb{Q})$, there is a

$$\beta \in N^p H^{2p+k}(X; \mathbb{Q})$$

such that

$$(\alpha, \beta)_X \neq 0$$

where $q = n - p - k$ and $(\bullet, \ \bullet)$ denotes the intersection number.

Since $p = 0$ is a trivial case, it suffices to prove two cases: 1). $p \ge 2$; 2). $p = 1$.

Case 1: $p \ge 2$.

Let E be an elliptic curve and

$$Y = X \times E.$$

Also let

$$P : Y \to X$$

be the projection. Let $a, a' \in H^1(E; Q)$ be a standard basis, i.e.

$$a \cup a' = 1, a \cup a = a' \cup a' = 0.$$

Let

$$\Lambda \subset H^n(X; \mathbb{Q}) \tag{4.8}$$

be a sub-Hodge structure of X of level k, containing α. Then

$$\Lambda \otimes H^1(E; \mathbb{Q}) \tag{4.9}$$

is the sub-Hodge structure of Y of level $k+1$ containing $\alpha \otimes a'$. Thus

$$\alpha \otimes a' \in M^p H^{2p+k+1}(Y). \tag{4.10}$$

Applying the Poincaré duality for sub-Hodge structures, we obtain

$$\theta \in M^p H^{2p+k+1}(Y)$$

such that

$$(\alpha \otimes a', \theta)_Y \neq 0. \tag{4.11}$$

We can choose θ to be generic in $M^p H^{2p+k+1}(Y)$, i.e. it has Hodge level

$$k+1.$$

Next we turn the level from Hodge to geometric (a key step). We consider the Gysin homomorphism

$$P_* : H^\bullet(Y;\mathbb{C}) \quad \to \quad H^{\bullet-2}(X;\mathbb{C}) \tag{4.12}$$

If n is odd,

$$M^{p-1}H^{2p+k-1}(X)$$

is non-zero because it contains a non-zero cycle $u^{\frac{n-1}{2}}$. If n is even, it contains subspace $H^1(X;\mathbb{Q})u^{\frac{n}{2}-1}$ which is also non-zero by the assumption. Hence the

$$im(P_*) = M^{p-1}H^{2p+k-1}(X) \neq 0.$$

Since θ is generic in the linear space $M^pH^{2p+k+1}(Y)$, $P_*(\theta) \neq 0$. Notice

$$2p+k-1 = n-1$$

is less than middle dimension of X. Applying the hard Lefschetz theorem on X, $P_*(\theta) \cup u$ is non-zero in

$$M^pH^{2p+k+1}(X).$$

Let

$$i : X_{n-1} \quad \hookrightarrow \quad X \tag{4.13}$$

be the inclusion map of a smooth hyperplane section X_{n-1}. Then by lemma 6.2, [7]

$$i_* \circ i^*(P_*(\theta)) = P_*(\theta) \cup u. \tag{4.14}$$

Because $P_*(\theta) \cup u$ is non-zero, neither is

$$i^*(P_*(\theta)).$$

Notice

$$i^*(P_*(\theta)) \in M^{p-1}H^{2p+k-1}(X_{n-1}). \tag{4.15}$$

and

$$dim(X_{n-1}) = n-1, X_{n-1} \in Corr_1(\mathbb{C}).$$

By the induction, the Hodge level turns to geometric level, i.e.

$$i^*(P_*(\theta)) \in N^{p-1}H^{2p+k-1}(X_{n-1}). \tag{4.16}$$

Hence by the formula (4.14)

$$P_*(\theta) \cup u \in N^pH^{2p+k+1}(X_{n-1}). \tag{4.17}$$

(The Gysin map i_* preserves the level for both filtrations). Applying the Grothendieck duality, we obtain that

$$P_*(\theta) \in N^{p-1}H^{2p+k-1}(X). \tag{4.18}$$

In the following we prove Lemma 2.4, i.e

$$\theta \in N^p H^{2p+k+1}(Y) \tag{4.19}$$

i.e. $P_*(\theta), \theta$ should have the same level in their own ambient varieties.

Proof of Lemma 2.4: First we use singular cycles. Let T'_θ be a singular cycle on Y representing θ. By Lemma A.4, there is another singular cycle T''_θ on Y finite to X such that

$$T''_\theta = T'_\theta + dK \tag{4.20}$$

where the "finite " is defined in Definition A.1 as being finite-to-one as a map. Because $T''_\theta \to X$ is finite, by taking multiple barycentric subdivisions,

$$P : Y \to X$$

is diffeomorphic to its image restricted to the interior of each $n+1$-simplex of T''_θ. Assume the push-forward $P_*(T''_\theta)$ is again a singular cycle of dimension $n+1$ in X. By the formula (4.18), the cohomology class of $P_*(T''_\theta)$ is

$$P_*(\theta) \in N^{p-1} H^{2p+k-1}(X) \tag{4.21}$$

(which is geometrically leveled). Hence we have formula

$$P_*(T''_\theta) = T_a + dL, \tag{4.22}$$

where T_a is supported in an algebraic cycle Z' of codimension $p-1$, and L is a singular chain. Now we let

$$T_\theta = T''_\theta - dL \times \{e\}. \tag{4.23}$$

where $e \in E$ is a point. Because $P : T_\theta \to X$ is again finite-to-one on the an Euclidean open set, the singular cycle T_θ must lie in the algebraic set

$$Z = Z' \times E$$

of codimension $p-1$. The following graph summarizes what we obtained

$$\begin{array}{ccccc}
\textit{Spaces} & \textit{Cohomology} & \textit{Singular Cycles} & & \textit{Algebraic subsets} \\
---- & ------ & ---------- & & ---------- \\
Y & \theta & T_\theta & \subset & Z \\
\downarrow_P & \downarrow_{P_*} & \downarrow_{P_*} & & \downarrow_P \\
X & P_*(\theta) & T_a & \subset & Z'
\end{array} \tag{4.24}$$

To continue, we let $\tilde{Z}$ be the smooth resolution of Z. We have the following composition map j:

$$j : \tilde{Z} \quad \to \quad Z \quad \to \quad Y. \tag{4.25}$$

By Corollary 8.2.8, [1], there is an exact sequence

$$H^{k+3}(\tilde{Z};\mathbb{Q}) \overset{j_*}{\to} H^{2p+k+1}(Y;\mathbb{Q}) \quad \to \quad H^{2p+k+1}(Y-Z;\mathbb{Q}). \tag{4.26}$$

Since non-zero T_θ is supported on Z, Proposition 3.1, [7] asserts the cohomology class θ is in the kernel of

$$H^{2p+k+1}(Y;\mathbb{Q}) \quad \to \quad H^{2p+k+1}(Y-Z;\mathbb{Q}). \tag{4.27}$$

Hence there is a class

$$\theta_{\tilde{Z}} \in M^1 H^{k+3}(\tilde{Z}) \tag{4.28}$$

such that

$$j_*(\theta_{\tilde{Z}}) = \theta. \tag{4.29}$$

In the following we discuss a couple of cases for the class $\theta_{\tilde{Z}}$ on $\tilde{Z}$, whose dimension is

$$p+k+2 = 2-p+n$$

and it lies in $Corr_1(\mathbb{C})$.

(a) If the coniveau $p > 2$, then $k+4 < dim(Z) < n$. By the assumption of the induction

$$\theta_{\tilde{Z}} \in N^1 H^{3+k}(\tilde{Z}). \tag{4.30}$$

(Hodge level turns into geometric level). By [7], the geometric level of cycle classes under the Gysin homomorphism j_* must be preserved. Thus we obtain that,

$$j_*(\theta_{\tilde{Z}}) = \theta \in N^p H^{2p+k+1}(Y). \tag{4.31}$$

This proves Lemma 2.4 for case (a).

(b) If $p = 2$, then Z has dimension $n = k+4$. Thus $k+3$ is not a middle dimension for $\tilde{Z}$. Then we consider the Grothendieck duality

$$u : M^1 H^{k+3}(\tilde{Z}) \quad \to \quad M^2 H^{k+5}(\tilde{Z}) \tag{4.32}$$

where u is a hyperplane section class represented by the hyperplane V. Let

$$l : V \cap \tilde{Z} \hookrightarrow \tilde{Z}$$

be the inclusion map. Then

$$l^*(\theta_{\tilde{Z}}) \tag{4.33}$$

is a class on $V \cap \tilde{Z}$ which must be $\mathcal{M}_{n-3}$ leveled. Since $V \cap \tilde{Z}$ has dimension

$$k+3 = n-1^{\mathrm{II}}$$

and $V \cap \tilde{Z} \in Corr_1(\mathbb{C})$, we apply the induction to obtain that

$$l^*(\theta_{\tilde{Z}}) \tag{4.34}$$

[II] This shows that the lowest n for our method is 4. Our method does not apply to the case $n = 2$ or 3.

is $\mathcal{N}_{n-3}$ leveled in $V \cap \tilde{Z}$. Notice

$$l_* \circ l^*(\theta_{\tilde{Z}}) = \theta_{\tilde{Z}} \cup u \tag{4.35}$$

Hence $\theta_{\tilde{Z}} \cup u$ is $\mathcal{N}_{n-3}$ leveled in $\tilde{Z}$. Now we use Grothendieck duality which guarantee

$$\theta_{\tilde{Z}}$$

is $\mathcal{N}_{n-2}$ leveled in $\tilde{Z}$ (Hodge level turns into geometric level).

In terms of the coniveau, it says

$$\theta_{\tilde{Z}} \in N^1 H^{3+k}(Z).$$

Next we repeat the part (a) to complete the proof for Lemma 2.4.

At last we use "descending construction" – Lemma 2.5 to extract a lower geometric level from the higher geometric level, θ.

Proof of Lemma 2.5: Applying Lemma 2.4, we obtain a non-empty algebraic set W of dimension at most $p+k+1$ such that θ is Poincaré dual to a singular cycle inside of W, i.e.

$$supp(T_\theta) \subset W. \tag{4.36}$$

Applying the Künneth decomposition, the singular cycles T_θ must be in the form of

$$\beta \otimes b + \beta' \otimes b' + \varsigma + dK \tag{4.37}$$

where β, β' represent cycles in X, whose cohomology have Hodge levels k, b, b' represent a, a', dK is exact and ς is the sum of currents in the form $\zeta \otimes c$ with $deg(c) = 0, 2$. We should note that the decomposition is in Lebesgue currents, where the tensor product is the current's tensor product. So we apply the real intersection theory ([8]). To prepare for the intersection of currents, let E, X be equipped with de Rham data and $X \times E$ be equipped with the product de Rham data. Choose a singular cycle b'' in E such that the intersection satisfy

$$[b'' \wedge b'] = 0, [b'' \wedge b] = \{e\}$$

where $e \in E$ (i.e. b'' and b' have the same cohomology.) Then we use the intersection in real intersection theory to obtain that the currents' intersection

$$[(X \otimes b'') \wedge T_\theta] = [(X \otimes b'') \wedge dK] + \beta \otimes \{e\} \tag{4.38}$$

is a Lebesgue current supported on W ((4.38) requires Leibniz rule in II, [8]). Formula (4.38) has implications in two different aspects of the cohomology.

(1) **Level in level filtration.** Let $\tilde{W}$ be a smooth resolution of the scheme W. We obtain the diagram

$$\begin{array}{ccccc} H^{k+2}(\tilde{W}; \mathbb{Q}) & \stackrel{\mu_*}{\to} & H^{2p+k+2}(Y; \mathbb{Q}) & \stackrel{R}{\to} & H^{2p+k+2}(Y - W; \mathbb{Q}) \\ & \nu_* \searrow & \downarrow P_* & & \\ & & H^{2p+k}(X; \mathbb{Q}) & & , \end{array} \tag{4.39}$$

where the top sequence is the Gysin exact sequence, and ν_*, which is a Gysin homomorphism, is the composition of Gysin homomorphism μ_*, P_*. Now we consider (4.39). Notice T_θ is supported on W. Then the real intersection theory implies that the intersection

$$[(X \otimes b'') \wedge dK] + \beta \otimes \{e\}$$

is also supported on W. Hence the current $\beta \otimes \{e\}$ is supported on W. By Proposition 3.1, [7], cohomology of $\beta \otimes \{e\}$, denoted by

$$\langle \beta \otimes \{e\} \rangle$$

is in the kernel of R. Hence it has a pre-image

$$\phi \in H^{k+2}(\tilde{W}; \mathbb{Q}).$$

Because μ_* is an algebraic correspondence, ϕ can be chosen to have Hodge level k(this is due to the strictness of the morphism of Hodge structures). Since the $dim(\tilde{W}) = p < n$, the inductive assumption says the Hodge level is the geometric level. The Gysin image $\nu_*(\phi)$ then also has geometric level k. The class

$$P_* \langle \beta \otimes \{e\} \rangle$$

is represented by the current β. Hence β representing the class $\nu_*(\phi)$ is $\mathcal{N}_k$ leveled.

(2) **Toplological intersection number.** On the other hand the intersection number by (2.8) satisfies

$$(\alpha \otimes a', \theta)_Y = (\alpha, \langle \beta \rangle)_X \neq 0. \tag{4.40}$$

Part (1), (2) conclude β represents the cohomology class of geometric level k and satisfies the condition in Lemma 2.5. □

Case 2: Coniveau $p = 1$.

Now we deal with the case when $p = 1$. In this case we already have all Proposition 2.2 for $p \neq 1$. We consider $\alpha \in M^1 H^n(X)$ where $n = dim(X)$ is any whole number. Then as before E is an elliptic curve, $Y = X \times E$ and $a, a' \in H^1(E; \mathbb{Q})$ form a standard basis in the cohomology ring. In the following we'll use the projection $P : Y \to X$, but on a different type of cycles. First

$$\alpha \otimes 1 \in M^1 H^n(Y). \tag{4.41}$$

Let $\theta \in M^2 H^{n+2}(Y)$ be its dual, i.e. a generic vector in the cohomology that the intersection number $(\alpha \otimes 1, \theta) \neq 0$. By the Proposition 2.2 for $p \neq 1$,

$$M^2 H^{n+2}(Y) = N^2 H^{n+2}(Y)$$

(geometric coniveau is 2) we obtain that

$$\theta \in N^2 H^{n+2}(Y).$$

Now we apply the Künneth decomposition for classes,

$$\theta = \beta \otimes \omega + \beta'' \otimes a + \beta' \otimes a' + \gamma \otimes 1 \tag{4.42}$$

where ω is the fundamental class of a point of E. Because $P_*(\theta)$ and θ will have the same geometric level, $P_*(\theta)$ lies in

$$N^1H^n(X).$$

Looking back to the formula (4.42), $P_*(\theta) = \beta$. This shows

$$\beta \in N^1H^n(X).$$

On the other hand, we see that

$$(\alpha \otimes 1, \theta)_Y = (\alpha, \beta)_X \neq 0. \tag{4.43}$$

Hence the intersection form on

$$M^1H^n(X) \times N^1H^n(X)$$

is perfect. Hence

$$M^1H^n(X) = N^1H^n(X).$$

We complete the proof for $X \in Corr_1(\mathbb{C})$.

At last we choose an arbitrary n-dimensional smooth projective variety X that may not be in

$$Corr_1(\mathbb{C}).$$

Let $\alpha \in M^pH^{2p+k}(X; \mathbb{Q})$ be non-zero, where $2p + k = n$. Then

$$\alpha \otimes 1 \in M^pH^{2p+k}(X \times E)$$

is non-zero. Since $X \times E \in Corr_1(\mathbb{C})$, Proposition 2.2 holds on $X \times E$. So there is a cycle

$$\theta \in N^{p+1}H^{2p+k+2}(X \times E)$$

such that

$$(\alpha \otimes 1, \theta) \neq 0. \tag{4.44}$$

Now we use Künneth decomposition to express

$$\theta = \beta \otimes \omega + \beta'' \otimes a + \beta' \otimes a' + \gamma \otimes 1 \tag{4.45}$$

as in (4.42). Recall $P : X \times E \to X$ is the projection. In general Gysin homomorphism preserves the geometric level. Since P_* is the Gysin homomorphism,

$$P_*(\theta) = \beta \in N^pH^{2p+k}(X \times E). \tag{4.46}$$

On the other hand,

$$(\alpha \otimes 1, \theta) = (\alpha, \beta) \neq 0. \tag{4.47}$$

We complete the proof for Proposition 2.2.

A Support of the projection

In this Appendix, we study the supports of cellular cycles in a Cartesian product.

Let $\mathcal{X}$ be a compact manifold of dimension n. We use the following setting in algebraic topology. A p-singular simplex S consists of three elements: a p dimensional polyhedron Δ^p in $\mathbb{R}^v$, an orientation of $\mathbb{R}^v$, and a C^∞ map f of $\mathbb{R}^v$ to X. A chain is a linear combination of singular simplexes. The support $|S|$ of S is the image of S in X. A point in S is a point in $|S|$.

Let $\mathcal{Y}$ be another compact manifold of dimension m. Let

$$\mathcal{P} : \mathcal{Y} \times \mathcal{X} \quad \to \quad \mathcal{X} \tag{A.1}$$

be the projection.

Definition A.1. *Let σ be a C^∞ p-singular simplex of $\mathcal{Y} \times \mathcal{X}$. Let a be an interior point of σ. If*

$$\mathcal{P}^{-1} \circ \mathcal{P}(a) \cap \sigma$$

is a finite set, we say σ is finite at a. If σ is finite at all interior points of σ, we say σ is finite to $\mathcal{X}$. The chain is finite if each simplex in the chain is finite.

Proposition A.2. *For any C^∞ p-singular simplex σ in the coordinates chart of $\mathcal{Y} \times \mathcal{X}$ with $p \leq dim(\mathcal{X})$, there is barycentric subdiviosn (multiple times) of σ*

$$Sd(\sigma) = \sum_{finite\ i} C_i, \tag{A.2}$$

such that each simplex C_i is homotopic to a simplex finite to $\mathcal{X}$ and the homotopy is a constant on the $\partial Sd(\sigma)$

Proof. Let

$$\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^m \times \mathbb{R}^n$$

be the coordinate's charts for $\mathcal{Y}, \mathcal{X}, \mathcal{Y} \times \mathcal{X}$ respectively such that $p \leq n$. We would like to show that there is a multi barycentric subdivision to divide σ to a chain $\sum_{i=0}^N \sigma_i$ (a sum of smaller regular cells σ_i) such that there are homotopy σ_i' for each σ_i that is finite to $\mathbb{R}^n$, and boundary of σ_i' is the same as that of σ_i. We use a claim to construct such small simplex σ_i.

Claim A.3. *Let $g : \mathbb{R}^k \to \mathbb{R}^l$ be a C^∞ map with $l \geq k$. Let $q \in \mathbb{R}^k$ be a point. Then there is an open ball B of q and continuous map $g' : \mathbb{R}^k \to \mathbb{R}^l$. such that*

1) g is homotopically deformed to g' such that at all points on ∂B and the boundary D of the unit ball g is fixed under the homotopy,

2) g' in $B \backslash D$ is C^∞ and finite to one to its image in $\mathbb{R}^l$.

Proof. of Claim A. 3: Let θ_1, θ_2 be two analytic functions on $\mathbb{R}^k$ such that $\theta_1 = \epsilon, \theta_2 = 0$ define ∂B and D where ϵ is the radius of B. We consider the homotopy

$$(1-t)g + t(g + \theta_1\theta_2 h), \qquad t \in [0,1]. \tag{A.3}$$

where h is some C^∞ function. Thus g is homotopic to

$$\Big(g + \theta_1\theta_2 h\Big) \tag{A.4}$$

The determinant of a maximal minor of the differential J of

$$\Big(g + \theta_1\theta_2 h\Big) \tag{A.5}$$

is a polynomial in θ_1, θ_2 whose coefficients are C^∞ functions of h. Thus for a small ϵ, by choosing a suitable h, the determinant is non-zero for all points in B with $\theta_2 \neq 1$ and $\theta_1 \neq \epsilon$, i.e. the differential J has full rank. By mean value theorem

$$\Big(g + \theta_1\theta_2 h\Big)$$

is 1-to-1 to its image when restricted to $B \backslash D$.

It satisfies required conditions in Claim A. 3.

□

Now let f be the composition of

$$(\Delta^p)' \to \mathcal{Y} \times \mathcal{X} \to \mathcal{X}$$

where $(\Delta^p)'$ is a neighborhood of Δ^p . Next we cover $\bar{\Delta}^p$ with finitely many balls $B_i, i = 1, \cdots, l$ and the homotopy in Claim A .3 for each B_i. Consider the first open set B_1. Applying Claim A.3, f is homotopic to $f_1 : (\Delta^p)' \to \mathcal{X}$ such that the homotopy fix the map f on ∂B_1 and D and f is homotopic to f_1 which is finite-to-one on B_1. Then we repeat the homotopy from f_1 to f_2, from f_2 to f_3, $\cdots$, from f_{l-1} to f_l. Finally, we obtain a continuous map f_l which is finite-to-one in each B_i and is equal to f on D. Let C_i be the barycentric subdivisions obtained from the covering $B_i, i = 1, \cdots, l$. Then f_l is homotopy to f. We complete the proof.

□

Lemma A.4. *For any cellular cycle S in $Y \times X$, of dimension $p < dim(X)$, S is homopotic to a cycle finite to X.*

Proof. Let
$$S = \sum_i C_i \tag{A.6}$$
and each cell C_i satisfies Proposition A.2 with a homotopy h_i. Then there is synchronized homotopy with the same parameter $t \in [0,1]$ such that C_i is homotopic to another cell C_i' 1-to-1 to $\mathcal{X}$, but the boundary is fixed. Since the boundaries are not changed, this synchronized homotopy are glued together to yield a homotopy of the cycle S,
$$S' = \sum_i C_i'. \tag{A.7}$$
□

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